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THE NATIONAL TECHNICAL UNIVERSITY OF UKRAINE

“Igor Sikorsky Kyiv Polytechnic Institute”

HIGHER MATHEMATICS

Multivariable Calculus.

Vector Calculus.

<u>Elements of Theory</u>

Kyiv

Igor Sikorsky Kyiv Polytechnic Institute

2021

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This textbook is designed for students of the first year of technical university. It covers two content areas to be studied in the second semester: Differential and Integral Calculus of the Multivariable Function and Vector Calculus.

Each part contains basic mathematical conceptions and explains new mathematical terms. The most important concepts of Multivariable Calculus are explained and illustrated by figures and examples.

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Introduction

This textbook is designed for students of the first year of technical university. It covers three content areas to be studied in the second semester: Differential and Integral Calculus of the Multivariable Function and Vector Calculus.

The manual can be helpful for students who want to study the concepts of multivariable and vector calculus and apply them for solving some tasks from geometry and physics and so on.

Each part contains basic mathematical conceptions and explains new mathematical terms. The most important concepts are explained and illustrated by figures and examples.

The first three parts deal with the differential calculus of the multivariable function: concept of function of several variables, its limit, partial derivatives and application of them to finding relative and absolute extrema of functions of multiple variables.

The fourth and fifth sections are devoted to double and triple integrals. These parts include double integrals in polar coordinates triple integrals in cylindrical and spherical coordinates and various applications of multiple integrals.

The next two parts concerned with line integrals with respect to arc length and surface integral over the surface. Here students could find the methods of evaluation and application of such kinds of integrals.

In the eighth chapter it is introduced the concept of the scalar and vector fields. Here the notions of gradient, directional derivative, divergence and curl are defined.

The ninth section deals with line integrals of vector fields. Green’s Theorem, independence of path and the fundamental theorem of calculus for line integrals of vector fields are explained.

The next chapter is devoted to surface integrals of vector fields. Here Divergence Theorem and Stokes’ formula are discussed.

In the last part reader could find the physical application of Vector calculus: flux and circulation of the vector fields.

There are also five appendices presenting some of the topics previously studied, that could be helpful when reading this manual.

1. Function of Several Variables

1.1 The Concept of a Function of Several Variables

I. *Multivariable Function*

Previously, we have been studying the function of one variable. This is useful if the things you're working on can be described in two variables (one dependent variable and one independent variable). For example, the area S of a square depends only on the length of the side x:

$$S = x^2 .$$

However, in real world of physics and engineering most independent variables depend on more than one dependent variable. For example, the area S of a rectangle with sides of length x and y is expressed by formula:

$$S = xy .$$

That means: to each pair of values of x and y there exists a definite value of the area S. Here S is a function of two variables.

Multivariable calculus is useful considering that most natural phenomenon can be best described by functions of several variables and the tools from multivariable calculus can be used to describe their behavior.

Let us see a few more examples below.

1. The volume V of a rectangular parallelepiped with edges length x, y and z is expressed by the function of three variables

$$V = xyz .$$

2. Pressure of ideal gas can be written as

$$P = \frac{nRT}{V} ,$$

where, V is the volume in liters, n - the number of particles in moles, T - the temperature in Kelvin and R - the ideal gas constant (0.0821 liter atmospheres per moles Kelvin). Here we have a function of three variables V, n and T (we already know R because it is a constant).

3. If it is known the law of changing the temperature U of an object in 3D space ($\mathbb{R}^3$) with time then the temperature could be represented by the function of four variables

$$U = U(x, y, z, t).$$

Consider two sets: X from *n*-dimensional space ($X \in \mathbb{R}^n$) and $U \in \mathbb{R}$.

Definition. If it exists the correspondence between a set X and a set U such as each $\bar{x} = (x_1, x_2, \dots x_n) \in X$ is related to exactly one $u \in U$ then that relation is called a *function of several (nth) variable or multivariable (n-tiple variable) function.* It is denoted by $u = f(\bar{x})$ or $u = f(x_1, x_2, \dots x_n)$.

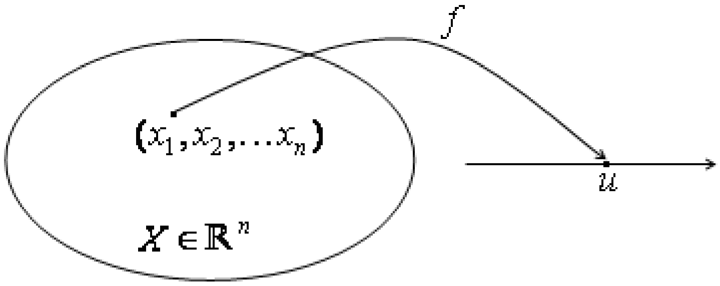


Figure 1.

The function of several variables could be represented by a formula (analytically) or by means of a table, where the value of the function corresponds to the set of independent variables.

In what follows, we will mainly consider double variable functions

$$u = f(x, y) \text{ or } z = f(x, y) \text{ for } (x, y) \in \mathbb{R}^2.$$

Most of definitions, theorems, explanations and examples will be made for the function of two variables. These ideas could be applied for the any multivariable function in the same fashion.

II. *Function of Two Variables and Its Geometric representation*

Let us consider the function of two variables $z = f(x, y)$, $(x, y) \in D \subset \mathbb{R}^2$.

According to the definition of the multivariable function, a function of two variables can be defined as a rule that assigns exactly one $z \in \mathbb{R}$ to each pair $(x, y) \in D \subset \mathbb{R}^2$.

The set $D \subset \mathbb{R}^2$ of points (x, y) where the function is defined is called its *domain of definition* of the function. It can be represented as a region on the plane. The *range* of the function is the set of its values $f(x, y)$, for all $(x, y) \in D \subset \mathbb{R}^2$.

If a function $z = f(x, y)$ is defined analytically, its domain D consists of all points (x, y) for which the formula makes sense.

Note. We mainly have to do with such domains as are parts of the plane bounded by lines. The lines bounding domain are called *boundary*. The points of the domain are called *interior points* if they do not lie on the boundary (Fig.2).

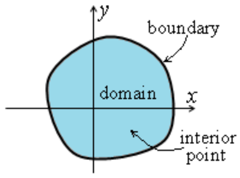


Figure 2.

A domain consisting of interior points is called *open* domain. If domain includes the points of boundary, we called it a *closed* domain (Fig.3).

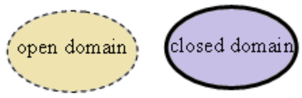


Figure 3.

Suppose $z = f(x, y)$ is a function of two variables, with domain $D \subset \mathbb{R}^2$. It is possible to graph any ordered pair (x, y) in the plane. With a function of two variables, each ordered pair (x, y) in the domain of the function corresponds to a real number z on the straight line. That means that at each point (x, y) erect a perpendicular to the xy-plane and on it lay off a segment equal to $f(x, y)$ (Fig.4).

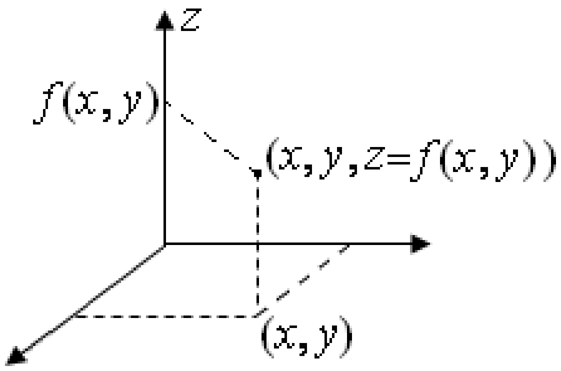


Figure 4.

The graph of a function of two variables is called a surface. It consists of ordered triples $(x, y, z = f(x, y))$. It is projected onto the xy-plane in the domain of definition of function.

Examples.

1. Let us consider the function $z = x^2 + y^2$.

The analytic expression $x^2 + y^2$ is meaningful for any pair $(x, y) \in \mathbb{R}^2$. Therefore, the domain of function is the entire xy-plane.

From analytic geometry we know that the graph of this function is paraboloid of revolution (Fig. 5).

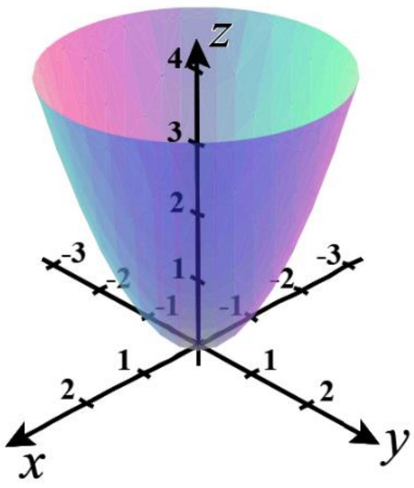


Figure 5.

2. Determine the domain of definition of the function $z = \dfrac{1}{x+y}$.

Since the division by zero is impossible operation, the following condition must be fulfilled:

$$x + y \neq 0 \quad \Leftrightarrow \quad y \neq -x.$$

Therefore, the domain of definition is xy-plane, except for the line $y = -x$ (Fig. 6).

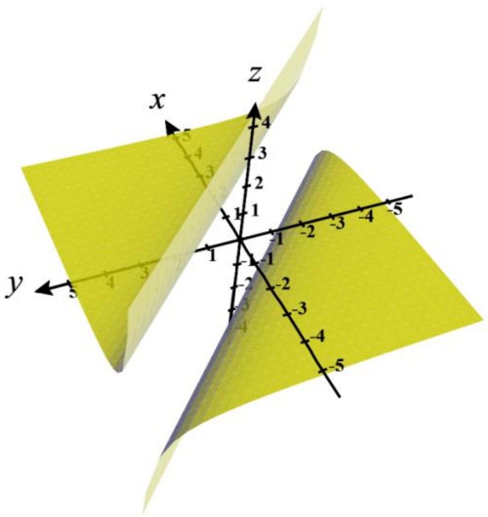


Figure 6.

3. For the function $z = \ln\left(1 - x^2 - y^2\right)$ find the domain of definition.

Logarithms are defined only for positive arguments. Hence, the following inequalities must be satisfied:

$$1 - x^2 - y^2 > 0 \Rightarrow x^2 + y^2 < 1.$$

Thus, the function exists only for the points (x, y) that lie inside the circle of radius 1 with centre at the origin, the circumference itself not included (Fig. 7).

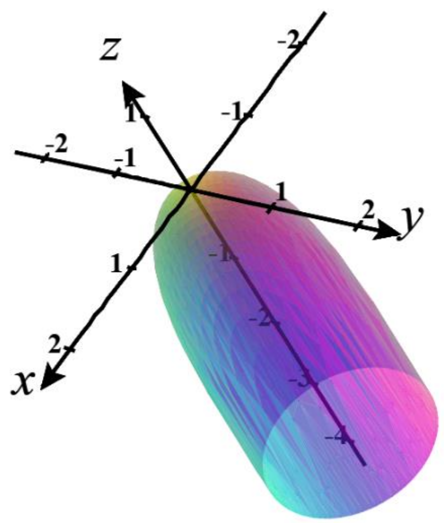


Figure 7.

Very useful tool for understanding the graph of a function of two variables is called level curve.

Consider the points of the domain D in which the function $z = f(x, y)$ has a fixed value c:

$$f(x, y) = c.$$

The totality of these points is a certain curve. We obtain a different curves while a different values of c are taken. These curves are called ***level curves.***

Level curves are projections in *xy*-plane of lines obtained at the intersection of the surface $z = f(x, y)$ with the plane $z = c$ (Fig.8).

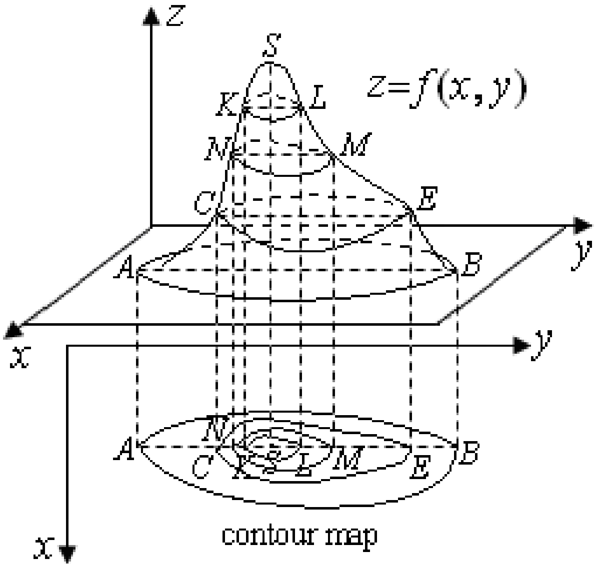


Figure 8.

One common example of level curves occurs in topographic maps of mountainous regions (Fig. 10). The level curves are curves of constant elevation. Notice that if you walk along one of these contour lines you neither ascend nor descend.

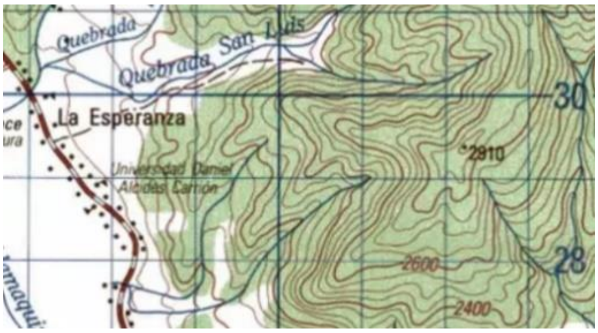


Figure 9.

Knowing the level curves, it is easy to study the character of the surface. A graph of the various level curves of a function is called a ***contour map.***

Another way of exploring the graph of a function of two variables are vertical traces. When level curves are always graphed in the *xy*-plane, vertical traces are graphed in the *xz*- or *yz*- planes. That means: ***traces of surfaces*** are curves that represent the intersection of the surface and the plane given by $x = a$ or $y = b$.

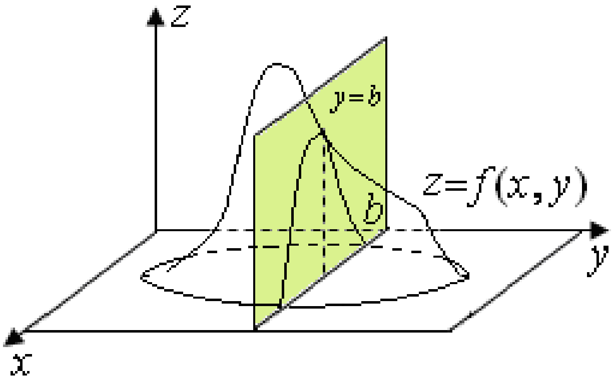


Figure 10.

1.2 Limit and Continuity of a Function of Several Variables

I. Let us consider the function of two variables $z = f(x, y)$ defined in some domain $D \subset \mathbb{R}^2$ and some definite point (x_0, y_0) in D or on its boundary.

Definition. The number A is called the *limit* of the function $f(x, y)$ as (x, y) tends to (x_0, y_0):

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = A,$$

if $\forall\, \varepsilon > 0\ \exists\, \delta(\varepsilon) > 0\ \forall\, (x, y) \in \left\{ 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta(\varepsilon) \right\}: \quad |f(x, y) - A| < \varepsilon$.

That means that the values of $f(x, y)$ can be made as "close as we like" to the number A by restricting points (x, y) in the domain D to be "sufficiently close" to (but different from) the point (x_0, y_0).

As you remember from the Single Variable Calculus there are only two ways of approaching x to x_0 along the x-axis: from the right or from the left. This leads to the one-sided limits at the point x_0: $\lim\limits_{x\to x_0+} f(x)$ and $\lim\limits_{x\to x_0-} f(x)$. Existence of the limit of a function of one variable means that the function must be approaching the same value as we take each of these directions in towards $x = x_0$.

For the function of two variables there are infinitely many different directions along which one point (x, y) can approach (x_0, y_0) (Fig. 11).

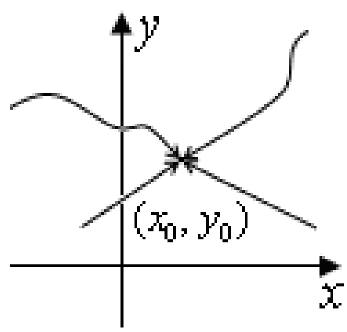


Figure 11.

Can be assumed that to find limit of the function of two variables we have to check an infinite number of directions and prove that the function is tends to the same value regardless of the path we are using to approach the point.

Note. This assumption make us to understand that finding limit is quite complicated task. But we can use it to prove that the limit does not exist.

Properties of Limits

Let $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = A$ and $\lim\limits_{(x,y)\to(x_0,y_0)} g(x,y) = B$, then

a) $\lim\limits_{(x,y)\to(x_0,y_0)} cf(x,y) = cA, \quad c \in \mathbb{R};$

b) $\lim\limits_{(x,y)\to(x_0,y_0)} (f(x,y) \pm g(x,y)) = \lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) \pm \lim\limits_{(x,y)\to(x_0,y_0)} g(x,y) = A \pm B;$

c) $\lim\limits_{(x,y)\to(x_0,y_0)} (f(x,y) \cdot g(x,y)) = \lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) \cdot \lim\limits_{(x,y)\to(x_0,y_0)} g(x,y) = A \cdot B;$

d) $\lim\limits_{(x,y)\to(x_0,y_0)} \left(f(x,y)\right)^{\alpha} = A^{\alpha}, \ \alpha \in (0,+\infty);$

e) $\lim\limits_{(x,y)\to(x_0,y_0)} \left(\dfrac{f(x,y)}{g(x,y)}\right) = \dfrac{\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)}{\lim\limits_{(x,y)\to(x_0,y_0)} g(x,y)} = \dfrac{A}{B}$, if $B \neq 0$.

We can apply these laws to finding limits of various functions.

Examples.

1. Calculate $\lim\limits_{(x,y)\to(1,2)} \dfrac{x+y}{2x+3y}$.

Since denominator is not equal 0 at point (1,2), we can use properties e), b) and a):

$$\lim_{(x,y)\to(1,2)} \frac{x+y}{2x+3y} = \frac{\lim\limits_{(x,y)\to(1,2)} (x+y)}{\lim\limits_{(x,y)\to(1,2)} (2x+3y)} = \frac{\lim\limits_{(x,y)\to(1,2)} x + \lim\limits_{(x,y)\to(1,2)} y}{2\lim\limits_{(x,y)\to(1,2)} x + 3\lim\limits_{(x,y)\to(1,2)} y} = \frac{1+2}{2\cdot 1 + 3\cdot 2} = \frac{3}{8}.$$

2. Determine whether the following limits exist $\lim\limits_{(x,y)\to(0,0)} \dfrac{x+y}{2x+3y}$.

The domain of the function $z = \dfrac{x+y}{2x+3y}$ consists of all points except for the point (0,0) (Fig. 12).

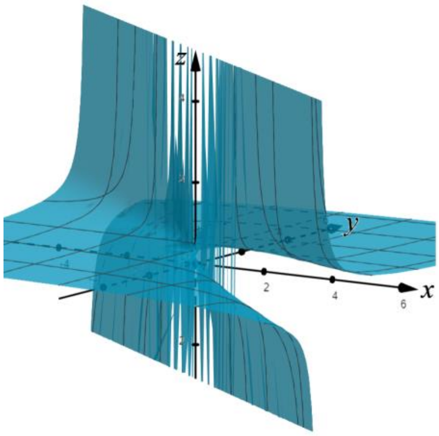


Figure 12.

To prove that the limit does not exist as (x,y) tends to (0,0), we show that the function takes different values along different lines passing through point (0,0).

Consider the line $y=0$ in the xy-plane. Substituting $y=0$ into the function gives

$$z(x,0) = \frac{x+0}{2x+3\cdot 0} = \frac{x}{2x} = \frac{1}{2}$$

for any value of x.

Next, consider the line $x=0$ in the xy-plane. Then

$$z(x,0)=\frac{0+y}{2\cdot 0+3y}=\frac{y}{3y}=\frac{1}{3}.$$

for any value of y.

Since $\lim\limits_{x\to 0} z(x,0)=\frac{1}{2}$, but $\lim\limits_{y\to 0} z(0,y)=\frac{1}{3}$, the limit fails to exist.

II. Let us consider a function $z=f(x,y)$, $(x,y)\in D\subset\mathbb{R}^2$ and the point $(x_0,y_0)\in D$.

Definition. The function $f(x,y)$ is continuous at the point (x_0,y_0) in its domain if the following conditions are satisfied:

1. $f(x_0,y_0)$ is defined;
2. $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ exists (here (x,y) approaches to (x_0,y_0) in arbitrary fashion all the while remaining in the domain of the function);
3. $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)=f(x_0,y_0)$.

A function continuous at each point of its domain D is called *continuous* on D. If a function is continuous at every point in the *xy*-plane, then it is called *continuous everywhere*.

If at some point (x_0,y_0) at least one of the conditions 1.-3. is not fulfilled, then this point is called *a point of discontinuity* of the function $z=f(x,y)$.

Continuous functions of two variables satisfy the usual properties well-known from the single variable calculus.

1. All the standard functions (power, exponential, trigonometric, logarithmic, inverse trigonometric functions) that we know to be continuous are still continuous even if we are plugging in more than one variable. We just need to watch out for division by zero, square roots of negative numbers, logarithms of zero or negative numbers, *etc*.

2. The sum of a finite number of continuous functions is a continuous function.

3. The product of a finite number of continuous functions is a continuous function.

4. The quotient of two continuous functions is a continuous function wherever the denominator is not equal zero.

5. If $g(x,y)$ is continuous at (x_0,y_0) and $f(u)$ is continuous at $u_0=g(x_0,y_0)$, then the composition of functions $f(x,y)=f(g(x,y))$ is continuous at (x_0,y_0).

6. If $f(x,y)$ is continuous at (x_0, y_0) and if $x(t)$ and $y(t)$ are continuous at t_0 with $x(t_0) = x_0$ and $y(t_0) = y_0$, then the composition $f(x(t), y(t))$ is continuous at t_0.

7. If a function is continuous in a closed bounded domain then it at least once reaches the maximum value and the minimum value.

8. If function is continuous in a closed bounded domain and assumes both positive and negative values, then there will be points inside the domain at which the function is equal zero.

2. Differential Calculus of a Function of Several Variables

2.1 Partial Derivatives of a Function of Several Variables

I. *Partial and Total Increment of a Function*

Let us consider a function of two variables $z = f(x, y)$ and the point $A(x, y)$ from the domain of function (Fig. 13).

Consider the line of intersection PP_1 of the surface $z = f(x, y)$ with the plane $y = const$ parallel to the xz-plane (Fig. 13). Along this curve z depends only on variable x. Increase the independent variable x by Δx (from point A to point C), then z will be increased too (from point P to point K). This increase is called ***the partial increment of*** z ***with respect to*** x and it is denoted by Δz_x, so that

$$\Delta z_x = f(x + \Delta x, y) - f(x, y). \qquad (2.1)$$

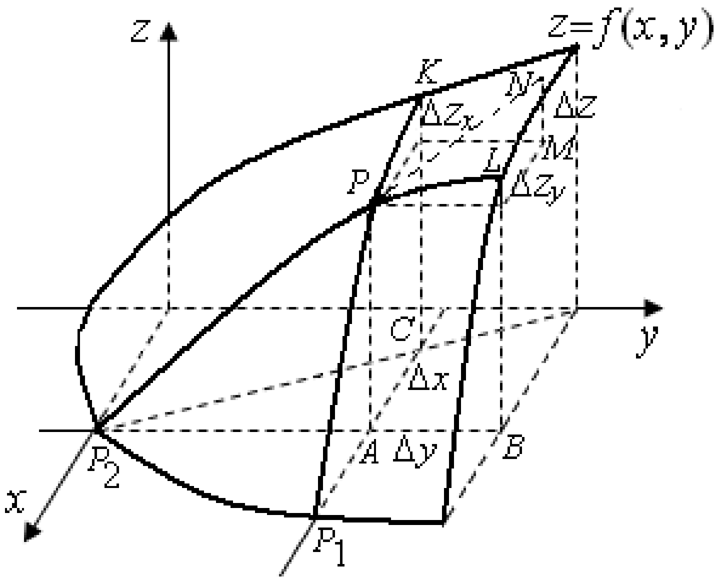


Figure 13.

Similarly, if $x = const$ and is increased y by Δy (from point A to point B), then z will be increased too (from point P to point L). Here we have ***the partial increment of*** z ***with respect to*** y:

$$\Delta z_y = f(x, y + \Delta y) - f(x, y). \qquad (2.2)$$

Finally, increasing both the argument x by Δx and the argument y by Δy (from point P to point N), we obtain an increment Δz (segment NM):

$$\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y). \tag{2.3}$$

This increment is called ***the total increment of the function*** z**.**

Note. Generally speaking the total increment is not equal to the sum of the partial increments

$$\Delta z \neq \Delta z_x + \Delta z_y .$$

II. ***Partial Derivatives***

When studying derivatives of functions of one variable, we defined derivative as an instantaneous rate of change of y as a function of x. We use the same idea defining derivatives for the function of two variables.

Definition. ***The partial derivative of the function*** $f(x,y)$ ***with respect to*** x is defined as

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

and denoted as $\frac{\partial f}{\partial x}$, f_x or $D_x f$.

In a like manner, ***the partial derivative of the function*** $f(x,y)$ ***with respect to*** y is defined as

$$\lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$$

and denoted as $\frac{\partial f}{\partial y}$, f_y or $D_y f$.

Thus, we have two different derivatives, since there are two different independent variables. Depending on which variable we choose, we can find different partial derivatives.

Taking derivatives of functions of two variables is done in the same manner as taking derivatives of a single variable. To compute $\frac{\partial f}{\partial x}$ we treat y as constant and then differentiate **with respect to** x as we've always done. Similarly, to compute $\frac{\partial f}{\partial y}$ we treat x as constants and then differentiate **with respect to** y.

The rules for computing partial derivatives coincide with the rules given for the function of one variable (sum, product, quotient and chain rules). We only have to remember with respect to which variable the derivative is taken.

Note. The partial derivatives of function of any number of values are determined in a same manner.

Example.

Given the function $z = x^3 + y^5 + x^6 y^2$, find the partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

According to rule for partial differentiation, we compute the partial derivative with respect to x we differentiate with respect to x and assume y is a constant

$$\frac{\partial z}{\partial x} = (x^3 + y^5 + x^6 y^2)'_x = 3x^2 + 6x^5 y^2$$

To compute the partial derivative with respect to y we differentiate with respect to y and assume x is a constant

$$\frac{\partial z}{\partial y} = (x^3 + y^5 + x^6 y^2)'_y = 5y^4 + 2x^6 y.$$

III. ***The Geometric Interpretation of the Partial Derivatives***

Consider the function of two variables $z = f(x, y)$. The graph of this function is some surface.

Let us draw the plane $y = const$. The corresponding trace of the surface is a curve PP_1 (Fig. 14). The limit $\lim\limits_{\Delta x \to 0} \frac{\Delta z_x}{\Delta x} = \frac{\partial z}{\partial x}$ is equal to the tangent of the angle α formed by the tangent line PR to this trace at the point P with the positive x-axis:

$$\frac{\partial z}{\partial x} = \tan \alpha .$$

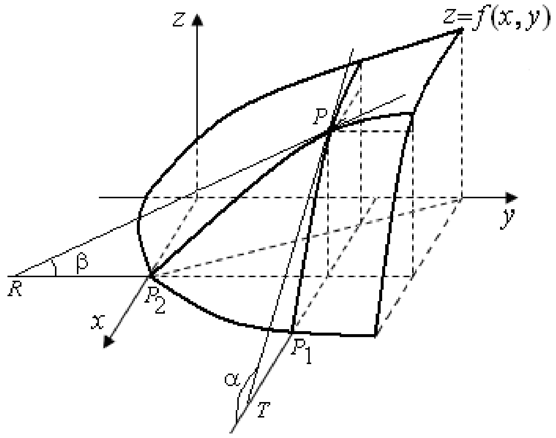


Figure 14.

Likewise, the partial derivative $\frac{\partial z}{\partial y}$ is equal to the tangent of the angle β of inclination of the tangent line to curve of intersection of the surface and the plane $x = const$:

$$\frac{\partial z}{\partial y} = \tan \beta .$$

2.2 Differential of a Function of Several Variables

Let us suppose that the function of two variables $z = f(x, y)$ has continuous partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$. Consider the total increment of function and let us express Δz in terms of partial derivatives.

Thus, add and subtract

$$\Delta z = f(x+\Delta x, y+\Delta y) - f(x, y) = \left| \begin{matrix} \text{add and subtract} \\ f(x, y+\Delta y) \end{matrix} \right| =$$

$$= [f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y)] + [f(x, y+\Delta y) - f(x, y)].$$

Applying the Mean-Value Theorem to the differences in the square brakets, we get

$$f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y) = \Delta x \frac{f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y)}{\Delta x} = \Delta x \frac{\partial f(x_1, y+\Delta y)}{\partial x},$$

where x_1 lies between x and $x+\Delta x$.

$$f(x, y+\Delta y) - f(x, y) = \Delta y \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y} = \Delta y \frac{\partial f(x, y_1)}{\partial y},$$

where y_1 lies between $y+\Delta y$ and y.

Hence,

$$\Delta z = \Delta x \frac{\partial f(x_1, y+\Delta y)}{\partial x} + \Delta y \frac{\partial f(x, y_1)}{\partial y}.$$

Since the partial derivatives are continuous,

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\partial f(x_1, y+\Delta y)}{\partial x} = \frac{\partial f(x, y)}{\partial x}$$

and

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\partial f(x, y_1)}{\partial y} = \frac{\partial f(x, y)}{\partial y}.$$

This equations could be written in the form

$$\frac{\partial f(x_1, y+\Delta y)}{\partial x} = \frac{\partial f(x, y)}{\partial x} + \gamma_1 \quad \text{and} \quad \frac{\partial f(x, y_1)}{\partial y} = \frac{\partial f(x, y)}{\partial y} + \gamma_2,$$

where the quantities γ_1 and γ_2 approach to zero as Δx and Δy approach to zero.

Therefore

$$\Delta z = \frac{\partial f(x,y)}{\partial x}\Delta x + \frac{\partial f(x,y)}{\partial y}\Delta y + \Delta x\gamma_1 + \Delta y\gamma_2 .$$

The sum of the later two terms of the rigth side is an infinitesimal of the high order relative to $\Delta x \to 0,\ \Delta y \to 0$.

The sum of the first two terms is the linear expression in Δx and Δy. For $\frac{\partial f(x,y)}{\partial x} \neq 0$ and $\frac{\partial f(x,y)}{\partial y} \neq 0$, this expression is ***the principal part of the increment***.

Definition. The linear part of the increment is called ***the total differential of the function*** $z = f(x,y)$ and denoted by

$$dz = \frac{\partial f(x,y)}{\partial x}dx + \frac{\partial f(x,y)}{\partial y}dy .$$

Example.

Given the function $z = \sin(2x + 3y) + x^3 y^2$, find the total differential.

Since,

$$\frac{\partial z}{\partial x} = 2\cos(2x + 3y) + 3x^2 y^2 \quad \text{and} \quad \frac{\partial z}{\partial y} = 3\cos(2x + 3y) + 2x^3 y ,$$

we get

$$dz = (2\cos(2x + 3y) + 3x^2 y^2)dx + (3\cos(2x + 3y) + 2x^3 y)dy .$$

Approximation by Total Differential

From the expression for total increment for the function $z = f(x,y)$ we have

$$f(x + \Delta x, y + \Delta y) = f(x,y) + \Delta z .$$

Since,

$$\Delta z \approx dz = \frac{\partial f(x,y)}{\partial x}\Delta x + \frac{\partial f(x,y)}{\partial y}\Delta y ,$$

we get the approximate formula

$$f(x + \Delta x, y + \Delta y) \approx f(x,y) + \frac{\partial f(x,y)}{\partial x}\Delta x + \frac{\partial f(x,y)}{\partial y}\Delta y$$

to within infinitesimals of the high order relative to Δx and Δy.

2.3 The Partial Derivatives of a Composite Function

Previously, when searching partial derivatives we have used chain rule, but there are several more complicated cases of composite functions, and we need to extend standart chain rule.

I. Let us consider the function $z = f(x, y)$, where x and y are functions of the independent variable t: $x = x(t)$ and $y = y(t)$.

If functions $x(t)$ and (t) are differentiable with respect to t and $z = f(x, y)$ is differentiable with respect to x and y, then $z = f(x(t), y(t))$ is differentiable with respect to t and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}.$$

Example. Compute $\dfrac{dz}{dt}$ for $z = x^2 y - e^{x+2y}$ if $x = t^2 + 1$ and $y = \cos t$.

Since,

$$\frac{\partial f}{\partial x} = 2xy - e^{x+2y}, \quad \frac{\partial f}{\partial y} = x^2 - 2e^{x+2y}$$

and

$$\frac{dx}{dt} = 2t, \quad \frac{dy}{dt} = -\sin t,$$

we obtain

$$\frac{dz}{dt} = (2xy - e^{x+2y}) \cdot 2t + (x^2 - 2e^{x+2y}) \cdot (-\sin t)$$

Technically we've computed the derivative. However, we should substitute in for x and y, that give us the expression of derivative in the terms of t:

$$\frac{dz}{dt} = (2(t^2+1)\cos t - e^{t^2+1+2\cos t}) \cdot 2t + ((t^2+1)^2 - 2e^{t^2+1+2\cos t}) \cdot (-\sin t) =$$

$$= 2t(t^2+1)\cos t - (t^2+1)^2 \sin t - 2e^{t^2+1+2\cos t}(t - \sin t).$$

Actually, the function z in this case is a function of only one variable t: $z = f(x(t), y(t)) = f(t)$, and it is easier to substitute in for x and y in the original function and compute the derivative with respect to t.

II. Assume now, that functions x and y are functions of two independent variables t and s: $x = x(t,s)$ and $y = y(t,s)$.

In this case,

$$z = f(x,y) = f(x(t,s), y(t,s))$$

is a composite function of two variables t and s.

If functions $z = f(x,y)$, $x = x(t,s)$ and $y = y(t,s)$ have continuous partial derivatives with respect to all their arguments.

Then

$$\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial t},$$

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s}.$$

Example. Compute $\frac{\partial z}{\partial t}$ and $\frac{\partial z}{\partial s}$ for $z = x^2 y - e^{x+2y}$ if $x = t^2 + s$ and $y = s^2 t$.

Let us find partial derivatives

$$\frac{\partial f}{\partial x} = 2xy - e^{x+2y}, \quad \frac{\partial f}{\partial y} = x^2 - 2e^{x+2y};$$

$$\frac{\partial x}{\partial t} = 2t, \quad \frac{\partial y}{\partial t} = s^2;$$

$$\frac{\partial x}{\partial s} = 1, \quad \frac{\partial y}{\partial s} = 2st.$$

Then,

$$\frac{\partial z}{\partial t} = (2xy - e^{x+2y})\cdot 2t + (x^2 - 2e^{x+2y})\cdot s^2 = 4(t^2+s)s^2t^2 + (t^2+s)^2 s^2 - 2(t+s^2)e^{t^2+s+2s^2t},$$

$$\frac{\partial z}{\partial s} = (2xy - e^{x+2y})\cdot 1 + (x^2 - 2e^{x+2y})\cdot 2st = 2(t^2+s)s^2t + 2st(t^2+s)^2 - (1+2st)e^{t^2+s+2s^2t}.$$

Differential for the function $z = f(x,y) = f(x(t,s), y(t,s))$ could be written as

$$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = \frac{\partial f}{\partial x}\left(\frac{\partial x}{\partial t}dt + \frac{\partial x}{\partial s}ds\right) + \frac{\partial f}{\partial y}\left(\frac{\partial y}{\partial t}dt + \frac{\partial y}{\partial s}ds\right) =$$

$$= \left(\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial t}\right)dt + \left(\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s}\right)ds = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial s}ds.$$

2.4 The Partial Derivatives of an Implicit Function

I. Let us consider the implicit function of one variable

$$F(x, y) = 0.$$

Assume that function $F(x, y)$ is continuous as a function of two variables and there exist partial derivative $\frac{\partial F}{\partial x}$, $\frac{\partial F}{\partial y}$ are continuous functions in some domain containing the point (x, y) whose coordinates satisfy equation $F(x, y) = 0$ and, additionally, $\frac{\partial F(x, y)}{\partial y} \neq 0$.

Then the function y of x has the derivative

$$\frac{dy}{dx} = -\frac{\frac{\partial F(x, y)}{\partial x}}{\frac{\partial F(x, y)}{\partial y}}.$$

Example.

Given the implicit function $x^3 + y^2 - xy = 0$, find $\frac{dy}{dx}$.

Here,

$$F(x, y) = x^3 + y^2 - xy,$$

$$\frac{\partial F(x, y)}{\partial x} = 3x^2 - y, \qquad \frac{\partial F(x, y)}{\partial y} = 2y - x.$$

Finally,

$$\frac{dy}{dx} = -\frac{3x^2 - y}{2y - x}.$$

II. Consider an equation of the form

$$F(x, y, z) = 0$$

If to each pair (x, y) there correspond the value of z (one or several) that satisfy this equation, then this equation implicitly defines function z of two variables x and y.

For example, the equation

$$x^2 + y^2 + z^2 - 1 = 0$$

implicitly defines the sphere of radius 1 centered in the origin of coordinates.

Let us assume that functions $F(x,y,z)$, $\dfrac{\partial F(x,y,z)}{\partial x}$, $\dfrac{\partial F(x,y,z)}{\partial y}$, $\dfrac{\partial F(x,y,z)}{\partial z} \neq 0$ are continuous as a functions of three variables in some domain containing the point (x,y,z) satisfying equation $F(x,y,z)=0$.

Then

$$\frac{\partial z}{\partial x} = -\frac{\dfrac{\partial F(x,y,z)}{\partial x}}{\dfrac{\partial F(x,y,z)}{\partial z}} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{\dfrac{\partial F(x,y,z)}{\partial y}}{\dfrac{\partial F(x,y,z)}{\partial z}}.$$

In the same manner, we can find partial derivatives for the implicit function of any number of variables.

Example. Find partial derivatives for the function $\sin(x+z) + x^2y^3z^4 = 0$.

Since,

$$\frac{\partial F(x,y,z)}{\partial x} = \cos(x+z) + 2xy^3z^4, \qquad \frac{\partial F(x,y,z)}{\partial y} = 3x^2y^2z^4,$$

$$\frac{\partial F(x,y,z)}{\partial z} = \cos(x+z) + 4x^2y^3z^3,$$

we get

$$\frac{\partial z}{\partial x} = -\frac{\cos(x+z) + 2xy^3z^4}{\cos(x+z) + 4x^2y^3z^3},$$

$$\frac{\partial z}{\partial y} = -\frac{3x^2y^2z^4}{\cos(x+z) + 4x^2y^3z^3}.$$

2.5 Partial Derivatives of Higher Orders

Assume that the function of two variables $z = f(x,y)$ is differentiable. Generally speaking, the partial derivatives $\dfrac{\partial z}{\partial x} = f'_x(x,y)$ and $\dfrac{\partial z}{\partial y} = f'_y(x,y)$ are functions of two variables x and y. Thus, we can differentiate each of them with respect to x and y again. As a result, we have four partial derivatives of the second order of the function of two variables.

The second partial derivatives are denoted as follows:

$\frac{\partial^2 z}{\partial x^2} = f''_{xx}(x, y)$, here f is differentiated twice with respect to x;

$\frac{\partial^2 z}{\partial x \partial y} = f''_{xy}(x, y)$, here f is differentiated first with respect to x and then the result is differentiated with respect to y;

$\frac{\partial^2 z}{\partial y \partial x} = f''_{yx}(x, y)$, here f is differentiated first with respect to y and then the result is differentiated with respect to x;

$\frac{\partial^2 z}{\partial y^2} = f''_{yy}(x, y)$, here f is differentiated twice with respect to y.

Higher-order partial derivatives calculated with respect to different variables, such as $f''_{xy}(x, y)$ and $f''_{yx}(x, y)$, are commonly called ***mixed partial derivatives*.**

Example. Find the second partial derivatives of the function $z = x^3 y^5 + x^4 - 3y^2$.

$$\frac{\partial z}{\partial x} = 3x^2 y^5 + 4x^3 \quad \Rightarrow \quad \begin{aligned} &\frac{\partial^2 z}{\partial x^2} = (3x^2 y^5 + 4x^3)'_x = 6xy^5 + 12x^2, \\ &\frac{\partial^2 z}{\partial x \partial y} = (3x^2 y^5 + 4x^3)'_y = 15x^2 y^4, \end{aligned}$$

$$\frac{\partial z}{\partial y} = 5x^3 y^4 - 6y \quad \Rightarrow \quad \begin{aligned} &\frac{\partial^2 z}{\partial y \partial x} = (5x^3 y^4 - 6y)'_x = 15x^2 y^4, \\ &\frac{\partial^2 z}{\partial y^2} = (5x^3 y^4 - 6y)'_y = 20x^3 y^3 - 6. \end{aligned}$$

Notice, that $f''_{xy}(x, y)$ and $f''_{yx}(x, y)$ are equal. Under certain conditions, this is always true.

Clairaut's Theorem

If a function $z = f(x, y)$ and its partial derivatives f'_x, f'_y, f''_{xy}, f''_{yx} are defined and continuous at the point $M(x, y)$ and in some neighborhood of it, then at this point

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}.$$

Clairaut's theorem states that it does not matter in which order we differentiate the functions, that is which variable goes first, then second. The proof of this theorem could be found, for example, in [1].

Second order derivatives are functions of two variables and could be differentiated both with respect to x and y.

Hence, we get eight partial derivatives of the third order.

$$
\begin{tikzcd}[column sep=small]
 & & & z=f(x,y) \arrow[dll] \arrow[drr] & & & & \\
 & \dfrac{\partial z}{\partial x} \arrow[dl] \arrow[dr] & & & & \dfrac{\partial z}{\partial y} \arrow[dl] \arrow[dr] & & \\
\dfrac{\partial^2 z}{\partial x^2} \arrow[d] \arrow[dr] & & \dfrac{\partial^2 z}{\partial x \partial y} \arrow[d] \arrow[dr] & & \dfrac{\partial^2 z}{\partial y^2} \arrow[d] \arrow[dr] & & \dfrac{\partial^2 z}{\partial y \partial x} \arrow[d] \arrow[dr] & \\
\dfrac{\partial^3 z}{\partial x^3} & \dfrac{\partial^3 z}{\partial x^2 \partial y} & \dfrac{\partial^3 z}{\partial x \partial y \partial x} & \dfrac{\partial^3 z}{\partial x \partial y^2} & \dfrac{\partial^3 z}{\partial y^2 \partial x} & \dfrac{\partial^3 z}{\partial y^3} & \dfrac{\partial^3 z}{\partial y \partial x^2} & \dfrac{\partial^3 z}{\partial y \partial x \partial y}
\end{tikzcd}
$$

Similarly, to Clairaut's theorem, we have

$$\frac{\partial^3 z}{\partial x^2 \partial y} = \frac{\partial^3 z}{\partial x \partial y \partial x} = \frac{\partial^3 z}{\partial y \partial x^2}$$

and

$$\frac{\partial^3 z}{\partial x \partial y^2} = \frac{\partial^3 z}{\partial y^2 \partial x} = \frac{\partial^3 z}{\partial y \partial x \partial y}.$$

Likewise, the fourth-order partial derivatives can be defined and so on. The *n*-th-order partial derivatives, where $n \geq 2$, are called the higher-order partial derivatives.

A corollary of Clairaut's theorem is that if the partial derivatives $\dfrac{\partial^n z}{\partial x^k \partial y^{n-k}}$ and $\dfrac{\partial^n z}{\partial y^{n-k} \partial x^k}$ are continuous, then

$$\frac{\partial^n z}{\partial x^k \partial y^{n-k}} = \frac{\partial^n z}{\partial y^{n-k} \partial x^k}.$$

Second and higher order partial derivatives for the function of three and more independent variables are defined analogously. A similar to Clairaut's theorem holds also for a function of any number of variables.

Example. Compute $\frac{\partial^5 u}{\partial x \partial y^2 \partial z^2}$ for the function $u = x^3 y^4 z^5$.

We have to differentiate first with respect to x, then twice with respect to y and then twice with respect to z

$$\frac{\partial u}{\partial x} = 3x^2 y^4 z^5 \Rightarrow \frac{\partial^2 u}{\partial x \partial y} = 12x^2 y^3 z^5 \Rightarrow \frac{\partial^3 u}{\partial x \partial y^2} = 36x^2 y^2 z^5 \Rightarrow \frac{\partial^4 u}{\partial x \partial y^2 \partial z} = 180x^2 y^2 z^4 \Rightarrow$$

$$\frac{\partial^5 u}{\partial x \partial y^2 \partial z^2} = 720x^2 y^2 z^3.$$

3. Application of Partial Derivatives

3.1 The Tangent Plane and the Normal Line to a Surface

Consider the function of two variables defined by an equation of the form

$$F(x, y, z) = 0.$$

This equation determines the surface in three-dimensional space.

Assume, that function F and its partial derivatives $\frac{\partial F}{\partial x}$, $\frac{\partial F}{\partial y}$, $\frac{\partial F}{\partial z}$ exist and are continuous, and at least one of them differs from zero, at some point $P(x_0, y_0, z_0)$.

Recall, that a straight line is a ***tangent*** to a surface at some point $P(x_0, y_0, z_0)$ if it is tangent line to some curve lying on the surface and passing trough this point (Fig. 15).

Since, there is infinite number of different curves lying on the surface passing through the point P, then there is infinitude of tangents to the surface passing through this point. All these tangent lines lie in one surface (the proof of this fact you can find in [1]).

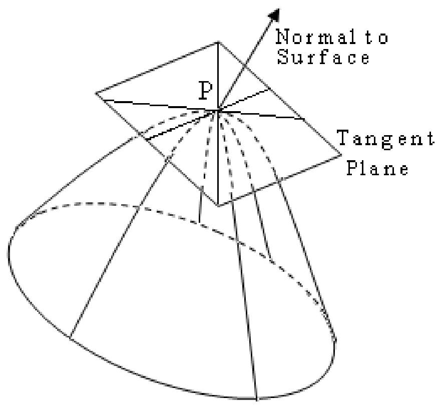


Figure 15.

This plane is called the ***tangent plane*** to the surface at the point P.

The straight line passed through the point P perpendicular to the tangent plane is called ***normal line*** to the surface.

The equation of the tangent plane for implicit function $F(x,y,z)=0$ at the point $P(x_0,y_0,z_0)$:

$$\left.\frac{\partial F}{\partial x}\right|_P (x-x_0)+\left.\frac{\partial F}{\partial y}\right|_P (y-y_0)+\left.\frac{\partial F}{\partial z}\right|_P (z-z_0)=0\,.$$

The equation of the normal line in this case is

$$\frac{(x-x_0)}{\left.\frac{\partial F}{\partial x}\right|_P}=\frac{(y-y_0)}{\left.\frac{\partial F}{\partial y}\right|_P}=\frac{(z-z_0)}{\left.\frac{\partial F}{\partial z}\right|_P}\,.$$

If the equation of the surface is $z=f(x,y)$ or $f(x,y)-z=0$ then

$$\frac{\partial F}{\partial x}=\frac{\partial f}{\partial x},\quad \frac{\partial F}{\partial y}=\frac{\partial f}{\partial y},\quad \frac{\partial F}{\partial z}=-1$$

and the equation of the tangent plane is

$$\left.\frac{\partial f}{\partial x}\right|_P (x-x_0)+\left.\frac{\partial f}{\partial y}\right|_P (y-y_0)-(z-z_0)=0\,.$$

The equation of the normal line is

$$\frac{(x-x_0)}{\left.\frac{\partial f}{\partial x}\right|_P}=\frac{(y-y_0)}{\left.\frac{\partial f}{\partial y}\right|_P}=\frac{(z-z_0)}{-1}\,.$$

Example. Find the tangent plane and normal line to the paraboloid $z=3-x^2-\frac{y^2}{4}$ at the point $P(1,-2)$.

Compute

$$z|_P=3-1^2-\frac{(-2)^2}{4}=1,\qquad \left.\frac{\partial f}{\partial x}\right|_P=-2x|_P=-2,\qquad \left.\frac{\partial f}{\partial y}\right|_P=-\left.\frac{y}{2}\right|_P=1\,.$$

Therefore, tangent plane (Fig. 16)

$$-2\cdot(x-1)+1\cdot(y+2)-(z-1)=0\Rightarrow$$

$$-2x+y-z+5=0;$$

normal line

$$\frac{x-1}{-2}=\frac{y+2}{1}=\frac{z-1}{-1}\,.$$

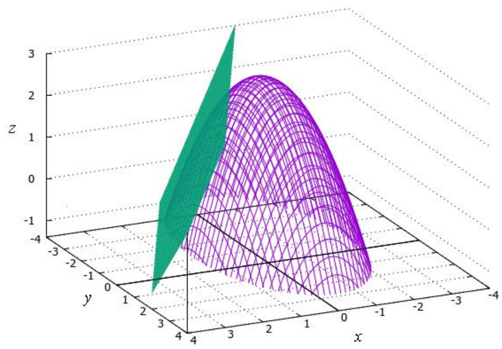


Figure 16.

3.2 Local Extrema of a Function of Two Variables

One of the most useful applications for derivatives in Single Variable Calculus is the determination of extrema (maxima and minima) values. Multivariable functions also have such points.

The definition of local (relative) extrema for functions of two variables is identical to that for functions of one variable.

Definition. Function $z = f(x, y)$ has a local maximum at the point $M(x_0, y_0)$ if

$$f(x_0, y_0) > f(x, y)$$

for all points (x, y) around to the point M (Fig. 17).

Definition. Function $z = f(x, y)$ has a local minimum at the point $M(x_0, y_0)$ if

$$f(x_0, y_0) < f(x, y)$$

for all points (x, y) around to the point M (Fig. 17).

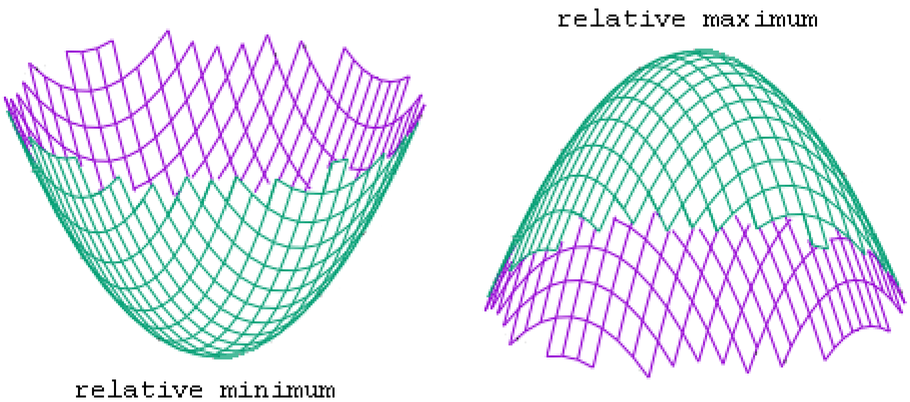


Figure 17.

Note. For functions of a single variable, at extremal points the tangent line is horizontal (perpendicular to y-axis) (Fig. 18). According to the definition of tangent line it gives us that derivative is equal zero.

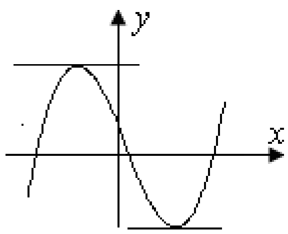


Figure 18.

For functions of two or more variables, the idea is the same. At relative extrema tangent plane is perpendicular to z-axis $(z = z_0 = f(x_0, y_0))$. It is possible when

$$f'_x(x_0, y_0) = 0 \text{ and } f'_y(x_0, y_0) = 0 .$$

Theorem 1. (Necessary Conditions for an Extremum)

If a function $z = f(x, y)$ has an extremum at the point $M(x_0, y_0)$ then at this point each partial derivative f'_x and f'_y are equal zero or do not exist.

This theorem is not sufficient for investigating the extremal points of the function. The points at which both f'_x and f'_y are equal zero or do not exist are called ***critical points*** of the function $z = f(x, y)$.

Theorem 2. (Sufficient Conditions for an Extremum)

Let a function $z = f(x, y)$ have continuous partial derivatives up to third order inclusive in a certain domain containing the critical point $M(x_0, y_0)$ ($f'_x(x_0, y_0) = 0$ and $f'_y(x_0, y_0) = 0$).

Determine the expression

$$\Delta(M) = f''_{xx}(x_0, y_0) \cdot f''_{yy}(x_0, y_0) - \left(f''_{xy}(x_0, y_0)\right)^2.$$

1. If $\Delta(M) > 0$ and $f''_{xx}(x_0, y_0) < 0$ then $f(x, y)$ has a maximum at the point $M(x_0, y_0)$.

2. If $\Delta(M) > 0$ and $f''_{xx}(x_0, y_0) > 0$ then $f(x, y)$ has a minimum at the point $M(x_0, y_0)$.

3. If $\Delta(M) < 0$ then $f(x, y)$ has neither maximum nor minimum at the point $M(x_0, y_0)$ (***minimax*** or ***saddle point***) (Fig. 18).

4. If $\Delta(M) = 0$ then the additional investigation is required (the test is inconclusive).

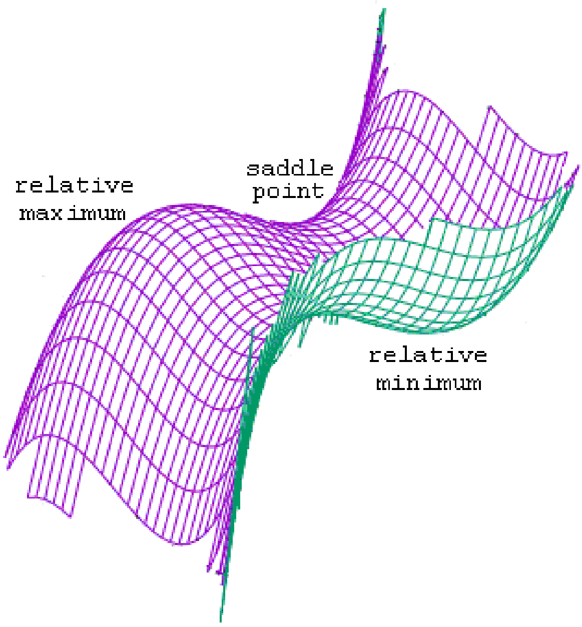


Figure 18.

Example. Find local extrema for the function $z = -2x^3 - 3y^2 + 6xy + 2$.

First, let us find critical points of the function.

Find partial derivatives

$$f'_x = -6x^2 + 6y \text{ and } f'_y = -6y + 6x$$

and solve the system of equations

$$\begin{cases} f'_x = 0, \\ f'_y = 0, \end{cases} \Rightarrow \begin{cases} -6x^2 + 6y = 0, \\ -6y + 6x = 0, \end{cases} \Rightarrow \begin{cases} x^2 - y = 0, \\ x = y, \end{cases} \Rightarrow \begin{cases} x^2 - x = 0, \\ x = y, \end{cases} \Rightarrow \begin{cases} x = 1, \ x = 0, \\ x = y. \end{cases}$$

Hence, critical points are $M_1(1,1)$ and $M_0(0,0)$.

Second, test each point for maximum and minimum.

Find the second order derivatives

$$f''_{xx} = -12x, \quad f''_{xy} = 6, \quad f''_{yy} = -6.$$

1. Investigate the character of the first critical point:

$$\Delta(M_1) = \left(-12x \cdot (-6) - (6)^2\right)\Big|_{x=1,\ y=1} = 36 > 0 \text{ and } f''_{xx}(M_1) = (-12x)\Big|_{x=1,\ y=1} = -12 < 0.$$

Therefore, at point $M_1(1,1)$ the given function has a maximum, namely:

$$z_{\text{loc max}} = 3.$$

2. Investigate the character of the second critical point:

$$\Delta(M_2) = \left(-12x \cdot (-6) - (6)^2\right)\Big|_{x=0,\ y=0} = -36 < 0.$$

Hence, at point $M_2(0,0)$ the function has a saddle point: $z = 2$.

The graph of the given function is presented on the Figure 19.

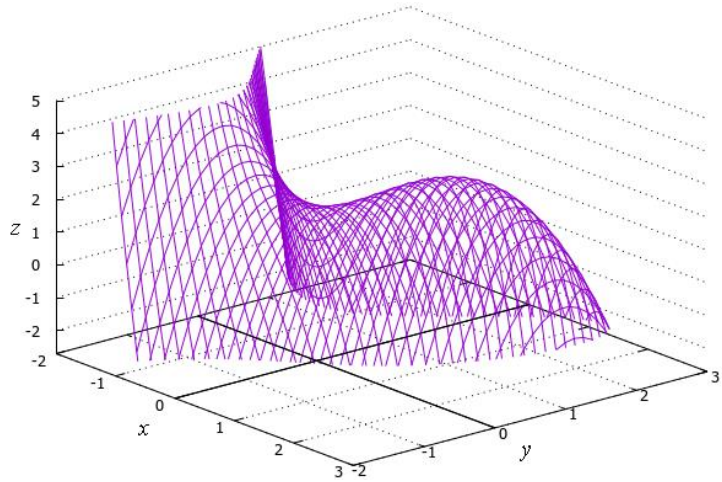


Figure 19.

3.3 Absolute Values of a Function of Two Variables in a Bounded Region

Let us suppose that our goal is to find the global maximum and minimum of a function in some closed bounded region $\overline{D} = D \cup \partial D$ in 2D-space. There are two types of points that can potentially be absolute maxima or minima. First, local extrema can be absolute extrema in the interior points of $\overline{D}$. Second, global extrema can occur at a boundary point of the region.

This leads to the algorithm of finding absolute extrema.

First step. Determine the critical points in the region D and calculate the corresponding critical values.

Second step. Evaluate the maximum and minimum value of the function on the boundary curves ∂D.

Third step. The global extrema values of the function occur at one of the values obtained in previous steps. The largest value of function corresponds to the absolute maximum and the smallest value corresponds to the global minimum.

Example. Evaluate the global extrema for the function $z = x^2 - y^2 + 2xy$ in the domain (Fig. 20)

$$\overline{D} = \{(x, y) : x^2 + y^2 \le 4,\ \ y \ge -1,\ \ y \ge -0{,}5x - 1\}.$$

First, start this off by finding all the critical points that lie inside the given region. To do this we have to find partial derivatives

$$f'_x = 2x + 2y\,,\ \ f'_y = -2y + 2x$$

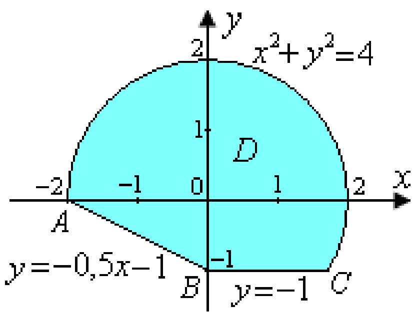


Figure 20.

and solve the system

$$\begin{cases} f'_x = 0, \\ f'_y = 0, \end{cases} \Rightarrow \begin{cases} 2x + y = 0, \\ -2y + 2x = 0, \end{cases} \Rightarrow \begin{cases} x = 0, \\ y = 0. \end{cases}$$

This critical point $(0{,}0)$ lie inside the region. Now we need to evaluate the function at this point

$$\underline{z(0{,}0) = 0}.$$

Next, we have to find the absolute extrema of the function along the boundary of D. That means that we need to investigate the global extrema of the function along each of the sides of the region.

The boundary of the region consists of three parts: segment AB, segment BC, arc of circle AC and "corner" points $A,\ B,\ C$.

Let us find coordinates of the "corner" points. It is easy to find coordinates of points $A(-2,0)$ and $B(0,-1)$. To find point C we need to solve a system of equations

$$\begin{cases} x^2 + y^2 = 4, \\ y = -1, \end{cases} \Rightarrow \begin{cases} x^2 = 3, \\ y = -1, \end{cases} \Rightarrow \begin{cases} x = \pm\sqrt{3}, \\ y = -1. \end{cases} \Rightarrow C(\sqrt{3},-1).$$

Evaluate the function at points $A,\ B,\ C$:

$$\underline{z(-2,0) = 4},$$

$$\underline{z(0,-1) = -1},$$

$$\underline{z(\sqrt{3},-1) = 2(1-\sqrt{3})}.$$

We can now look at segment AB defined by $y = -0{,}5x - 1,\ -2 \le x \le 0$.

Put $y = -0{,}5x - 1$ into the function $z = x^2 - y^2 + 2xy$ instead of y and get

$$z = x^2 - (-0{,}5x - 1)^2 + 2x(-0{,}5x - 1) = -0{,}25x^2 - 3x - 1.$$

Now the original function of two variables is a function of x only. We can use the methods from Single Variable Calculus to find relative extrema.

Find first order derivative

$$z' = -0{,}5x - 3.$$

Solve the equation

$$z' = 0 \ \Rightarrow \ -0{,}5x - 3 = 0 \ \Rightarrow \ x = -6.$$

Point $(-6,0)$ is out of segment AB and we do not use it further.

On the part BC, we have $y = -1,\ 0 \le x \le \sqrt{3}$.

Again, we put $y = -1$ into the original function and get

$$z = x^2 - (-1)^2 + 2x(-1) = x^2 - 2x - 1.$$

Since,

$$z' = 2x - 2$$

and

$$z' = 0 \;\Rightarrow\; 2x - 2 = 0 \;\Rightarrow\; x = 1.$$

Therefore, the relative extrema on segment BC is at $(1,-1)$ and

$$\underline{z(1,-1) = -2}.$$

The last part is an arc of circle of radius 2 centered at the origin: $x^2 + y^2 = 4$. To find extreme points we have to parameterize the circle. The parameterization is

$$\begin{cases} x = 2\cos t, \\ y = 2\sin t. \end{cases}$$

The value $t = \pi$ corresponds to point A and $t = -\frac{\pi}{6}$ corresponds to point C.

Therefore, $-\frac{\pi}{6} \le t \le \pi$.

Substitute these expressions into the function $z = x^2 - y^2 + 2xy$:

$$z = (2\cos t)^2 - (2\sin t)^2 + 2(2\cos t)(2\sin t) = 4\cos 2t + 4\sin 2t.$$

Since, the original functions are reduced to a function of single variable, we can determine the extrema on the circle using simple techniques.

Let us find first derivative

$$z' = -8\sin 2t + 8\cos 2t$$

and solve an equation

$$z' = 0 \;\Rightarrow\; -8\sin 2t + 8\cos 2t = 0 \;\Rightarrow\; \tan 2t = 1 \;\Rightarrow\; t = \frac{\pi}{8}.$$

Hence,

$$\begin{cases} x = 2\cos\frac{\pi}{8} = 2\sqrt{\frac{1}{2}\left(1 + \cos\frac{\pi}{4}\right)} = 2\sqrt{\frac{1}{2}\left(1 + \frac{\sqrt{2}}{2}\right)} = \sqrt{2 + \sqrt{2}}, \\ y = 2\sin\frac{\pi}{8} 2\sqrt{\frac{1}{2}\left(1 - \cos\frac{\pi}{4}\right)} = 2\sqrt{\frac{1}{2}\left(1 - \frac{\sqrt{2}}{2}\right)} = \sqrt{2 - \sqrt{2}}, \end{cases}$$

and

$$\underline{z\left(\sqrt{2 + \sqrt{2}}, \sqrt{2 - \sqrt{2}}\right) = 4\cos\frac{\pi}{4} + 4\sin\frac{\pi}{4} = 4\sqrt{2}}.$$

Finally, from the underlined values of z we choose the smallest and the largest values. The global maximum occurs at $(\sqrt{2+\sqrt{2}}, \sqrt{2-\sqrt{2}})$ and

$$z_{\max} = z(\sqrt{2+\sqrt{2}}, \sqrt{2-\sqrt{2}}) = 4\sqrt{2}\,.$$

The global minimum occurs at $(1,-1)$ and

$$z_{\min} = z(1,-1) = -2\,.$$

3.4 Conditional Extrema. Lagrange Multipliers

Finding global extrema values of the function on the boundary of the region can be complicated. If the boundary curve could be parameterized, then it is possible determine extrema, as seen in example. But if the boundary is a more complicated curve defined by implicit function $\varphi(x, y) = 0$, then the method of ***Lagrange multipliers*** can be useful for determining absolute extrema of the function on the boundary.

Let us optimize (find the minimum and maximum value of) the function $z = f(x, y)$ (the objective function) under the condition that x and y are connected by the equation $\varphi(x, y) = 0$ (the constraint function).

Construct an auxiliary function

$$F(x, y, \lambda) = f(x, y) + \lambda\varphi(x, y)\,.$$

Here the number λ is called a Lagrange multiplier.

The points at which a function $z = f(x, y)$ of two variables has conditional extrema subject to the constraint $\varphi(x, y) = 0$ are included among the points (x, y) determined by the first two coordinates of the solutions (x, y, λ) of the system of equations

$$\begin{cases} \dfrac{\partial f}{\partial x} + \lambda\dfrac{\partial \varphi}{\partial x} = 0, \\ \dfrac{\partial f}{\partial y} + \lambda\dfrac{\partial \varphi}{\partial y} = 0, \\ \varphi(x, y) = 0. \end{cases}$$

Example. Find the maximum value of function $z = xy$ if (x, y) is restricted to the ellipse $\frac{x^2}{8} + \frac{y^2}{2} = 1$ and $x \geq 0$, $y \geq 0$.

Construct an auxiliary function

$$F(x, y, \lambda) = xy + \lambda\left(\frac{x^2}{8} + \frac{y^2}{2} - 1\right).$$

Therefore, the system of equations that needs to be solved is

$$\begin{cases} y + \lambda \frac{x}{4} = 0, \\ x + \lambda y = 0, \\ \frac{x^2}{8} + \frac{y^2}{2} - 1 = 0. \end{cases}$$

Eliminating λ from first two equations we obtain

$$\begin{cases} \lambda = -\frac{4y}{x}, \\ \lambda = -\frac{x}{y}, \end{cases} \Rightarrow \quad -\frac{4y}{x} = -\frac{x}{y} \quad \Rightarrow \quad x^2 = 4y^2.$$

Substituting it into the last equation we get

$$\frac{4y^2}{8} + \frac{y^2}{2} - 1 = 0 \quad \Rightarrow \quad y^2 = 1 \quad \Rightarrow \quad y = \pm 1$$

and

$$x = \pm 2.$$

Since $x \geq 0$, $y \geq 0$, we obtain one point $(2,1)$. At the points of interception of ellipse $\frac{x^2}{8} + \frac{y^2}{2} = 1$ and coordinate axes $(2\sqrt{2},0)$ and $(0,2\sqrt{2})$ function $z = xy$ is equal zero. Therefore, the point $(2,1)$ is the point of conditional maximum and $z(2,1) = 2$.

4. Double Integral

4.1 The Concept of a Double Integral

I. Consider the solid $V \in \mathbb{R}^3$ such as its top base is defined by the continuous positive function $z = f(x, y)$, bottom base is a closed region D in xy-plane and side of solid whose directrix is the boundary of D and generators are parallel to z-axis (Fig. 21).

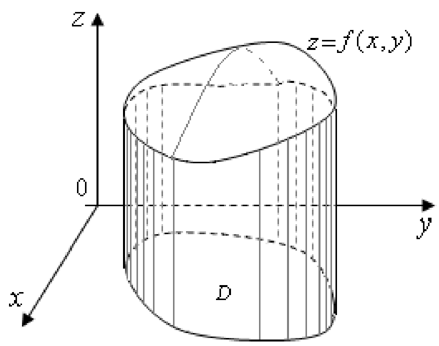


Figure 21.

Let us find the volume of this solid.

We will approximate the volume much as we approximated the area of curvilinear trapezoid.

First, we divide the region D into n parts (subdomains): D_1, D_2, $\ldots D_n$. Denote by ΔS_1, ΔS_2, $\ldots \Delta S_n$ the areas of subdomains.

Second, in each D_i we choose a point $P_i(x_i, y_i)$ and calculate $f(x_i, y_i)$ (Fig. 22).

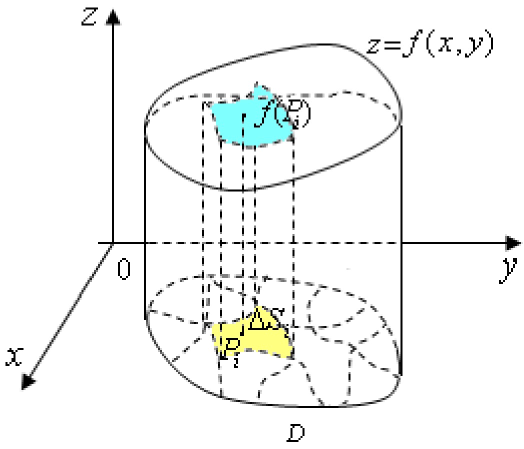


Figure 22.

Over each of small subdomain D_i we construct a cylindrical body whose height is given by $f(x_i, y_i)$. Volume of such small cylinder is

$$V_i = f(x_i, y_i)\Delta S_i .$$

Hence, the volume of V is approximately

$$V \approx \sum_{i=1}^{n} V_i = \sum_{i=1}^{n} f(x_i, y_i)\Delta S_i .$$

To get a better approximation we have to take n larger and larger and to get the exact volume we need to take the limit as n goes to infinity. In other words,

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i, y_i)\Delta S_i .$$

This looks like the definition of definite integral of a function of one variable. Indeed, this is the definition of a double integral or an integral of a function of two variables over the region D.

As $n \to \infty$ then subdomains D_i become smaller and smaller and diameter of such regions tends to zero.

Definition. If for continuous in D function $f(x, y)$, for any partition of the domain D such that $\max\limits_i diamD_i \to 0$ and for any choice of points P_i it exists the limit of integral sum, then that limit $\lim\limits_{\max\limits_i diamD_i \to 0} \sum\limits_{i=1}^{n} f(x_i, y_i)\Delta S_i$ is called ***the double integral or an integral of a function of two variables over*** D and denoted by

$$\iint\limits_D f(x, y)ds \quad \text{or} \quad \iint\limits_D f(x, y)dxdy .$$

The domain D is called the ***domain of integration*** and the function $f(x, y)$ is called ***integrable*** over D.

II. ***Properties of double integrals***

1. Let A, B be an arbitrary real numbers and $f(x, y)$, $g(x, y)$ be continuous in D functions. Then

$$\iint\limits_D (Af(x, y) \pm Bg(x, y))ds = A\iint\limits_D f(x, y)ds \pm B\iint\limits_D g(x, y)ds .$$

The proof of this theorem is the same as that of corresponding property for definite integral of the function of single variable.

2. If a domain D is divided into two domains D_1 and D_2 without common interior points, and a function $f(x, y)$ be continuous in all parts of D, then

$$\iint\limits_D f(x, y)ds = \iint\limits_{D_1} f(x, y)ds + \iint\limits_{D_2} f(x, y)ds .$$

Proof. The integral sum over D may be given in the form

$$\sum_D f(x_i, y_i)\Delta S_i = \sum_{D_1} f(x_i, y_i)\Delta S_i + \sum_{D_2} f(x_i, y_i)\Delta S_i ,$$

where the first sum contains terms that correspond to D_1, the second, those corresponding to D_2 (Fig.23).

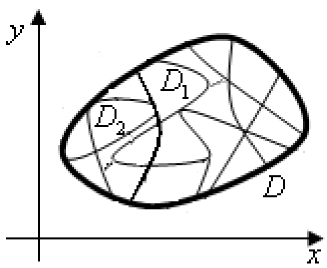


Figure 23.

Passing to the limit as $\max\limits_i diamD_i \to 0$, we obtain the result of the theorem. This theorem is obviously true for any number of terms.

4.2 Calculating Double Integral in Cartesian Coordinates

There are several cases, depending on the domain of integration, for calculating double integrals.

Case 1. Let a domain D lying in the xy-plane be such that any straight line parallel to y-axis and passing through an interior point of the domain, intersect the boundary of the domain only at two points (Fig. 24).

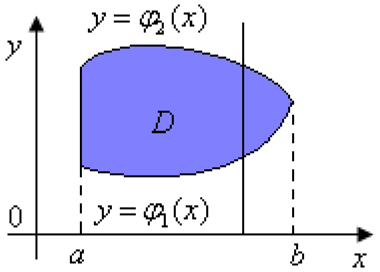


Figure 24.

In this case we can describe the domain D as

$D = \{(x, y) \mid \varphi_1(x) \le y \le \varphi_2(x), \; a \le x \le b\}$,

where the functions $y = \varphi_1(x)$ and $y = \varphi_2(x)$ are continuous on interval $[a,b]$.Such a domain is called *regular in the* y*-direction.*

Let the function $f(x, y)$ be continuous in D. Then

$$\iint_D f(x, y)ds = \int_a^b \left(\int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)dy \right) dx = \int_a^b dx \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)dy \,.$$

This is an example of an ***iterated integral***. One integrates with respect to y first. It gives us some function of x: $\Phi(x) = \int\limits_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)dy$. Then we integrate the result with respect to x: $\int\limits_a^b \Phi(x)dx$. The integrals with respect to y and x are called the inner and outer integrals, respectively.

Example. Calculate the integral $\iint\limits_D (x + 2y)ds$, where domain D is bounded by lines $y = 1$, $y = 2$, $x = 0$ and $x = 1$.

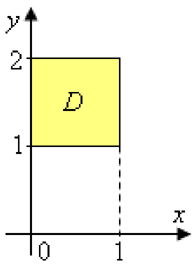


Figure 25.

Domain D is a rectangle (Fig. 25) and we can describe it as

$D = \{(x, y) \mid 1 \le y \le 2, \; 0 \le x \le 1\}$.

Therefore,

$$\iint_D (x + 2y)ds = \int_0^1 dx \int_1^2 (x + 2y)dy = \int_0^1 dx (xy + y^2)\Big|_1^2 =$$

$$= \int_0^1 ((2x + 4) - (x + 1))dx = \int_0^1 (x + 3)dx = \left(\frac{x^2}{2} + 3x \right)\Bigg|_0^1 = 3{,}5 \,.$$

Case 2. Let a domain D lying in the xy-plane be such that any straight line parallel to x-axis and passing through an interior point of the domain, intersect the boundary of the domain only at two points (Fig. 26).

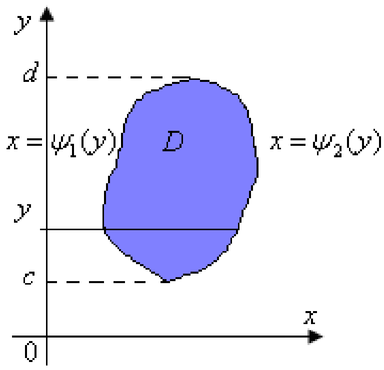


Figure 26.

In this case we can describe the domain D as

$$D = \{(x, y) \mid c \le y \le d,\ \psi_1(y) \le x \le \psi_2(y)\},$$

where the functions $\psi_1(y)$ and $\psi_2(y)$ are continuous on interval $[c, d]$. Such a domain is called *regular in the x-direction.*

Let the function $f(x, y)$ be continuous in D. Then

$$\iint_D f(x, y)ds = \int_c^d \left(\int_{\psi_1(y)}^{\psi_2(y)} f(x, y)dx \right) dy = \int_c^d dy \int_{\psi_1(y)}^{\psi_2(y)} f(x, y)dx.$$

First, we integrate with respect to x. As a result, we obtain a function of y: $\Psi(y) = \int_{\psi_1(y)}^{\psi_2(y)} f(x, y)dx$. Then we integrate with respect to y: $\int_c^d \Psi(y)dy$.

Examples.

1. Calculate the integral $\iint_D e^{\frac{x}{y}} dxdy$, where domain D is bounded by lines $y = x,\quad y = 0,\quad y = 1.$

The domain (Fig. 27) is regular in both x and y directions. Thus, we can choose any order of integration, but calculating of integral is possible when inner integral is with respect to x. Another choice gives us integral $\int e^{\frac{x}{y}} dy$, that have no primitive.

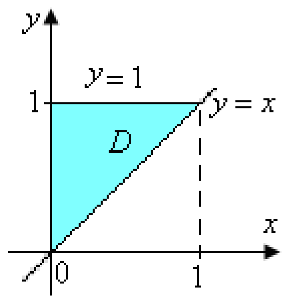


Figure 27.

Hence, we describe the domain D as

$$D = \{(x, y) \mid 0 \le y \le 1,\ 0 \le x \le y\},$$

$$\iint_D e^{\frac{x}{y}} dxdy = \int_0^1 dy \int_0^y e^{\frac{x}{y}} dx = \int_0^1 dy \left(ye^{\frac{x}{y}} \right)\Bigg|_0^y = \int_0^1 y(e - 1)dy = \frac{e - 1}{2}.$$

2. Calculate the integral $\iint\limits_D xds$, where domain D is bounded by lines $y = x, \quad y = 2x, \quad y + x = 6.$

Let's start by sketching the domain (Fig. 28). It is the triangle. We have to break the region up into two different pieces since the higher function is different depending upon the value of x. In this case the region would be given by $D = D_1 \cup D_2$, where

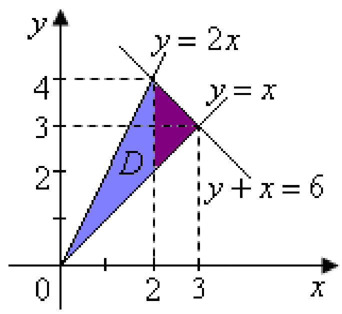


Figure 28.

$$D_1 = \{(x,y) \mid 0 \le x \le 2, \; x \le y \le 2x\},$$

$$D_2 = \{(x,y) \mid 2 \le x \le 3, \; x \le y \le 6 - x\}.$$

Domain D is the union of the two regions and we have to do two separate integrals, one for each of the regions.

$$\iint\limits_D xds = \iint\limits_{D_1} xds + \iint\limits_{D_2} xds = \int\limits_0^2 dx \int\limits_x^{2x} xdy + \int\limits_2^3 dx \int\limits_x^{6-x} xdy = \int\limits_0^2 dx (xy)\Big|_x^{2x} + \int\limits_2^3 dx (xy)\Big|_x^{6-x} =$$

$$= \int\limits_0^2 x^2 dx + \int\limits_2^3 (6x - 2x^2) dx = \frac{x^3}{3}\Bigg|_0^2 + \left(3x^2 - \frac{2x^3}{3}\right)\Bigg|_2^3 = 5.$$

4.3 Double Integral in Polar Coordinates

I. *Change of Variables in a Double Integral (General Case)*

Consider the double integral $\iint\limits_D f(x,y)dxdy$ over the domain D. Let us change of variables in the integral by setting

$$\begin{cases} x = x(u,v), \\ y = y(u,v), \end{cases} \tag{4.1}$$

where functions $x(u,v)$ and $y(u,v)$ are single-valuated, continuous and have continuous partial derivatives. In this case to each pair of values u, v corresponds a unique pair of values x, y and vice versa. It follows that with each point $M(x,y)$ in xy-plane there is uniquely associated a point $N(u,v)$ in uv-plane (Fig. 29).

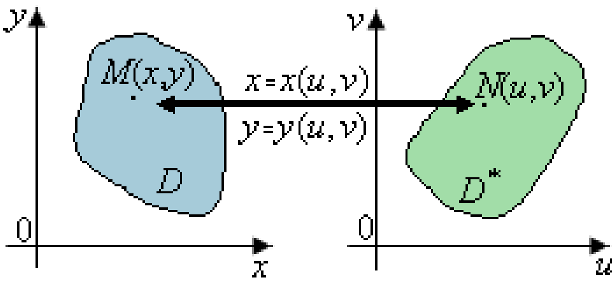


Figure 29.

If in the xy-plane a point describes a closed line bounding the domain D, then in the uv-plane a corresponding point will trace out a closed line bounding a certain domain D^*, and to each point of D^* there will correspond a point of D.

Thus, the formulas (4.1) give us *a one-to-one correspondence between the points of the domains* D *and* D^*.

It permits to reduce calculating of the double integral over a domain D to the computation of the double integral over a domain D^* by the formula

$$\iint\limits_{D} f(x,y)dxdy = \iint\limits_{D^*} f(x(u,v),y(u,v))\left|J(u,v)\right|dudv, \qquad (4.2)$$

where $J(u,v)=\begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} \neq 0$ is called the *Jacobian* or the *functional determinant of transformation*. Proof of this formula reader could find in [1] or [5].

II. *Double Integral in Polar Coordinates*

The transformation from rectangular coordinates to polar coordinates is a special case of change of variables in a double integral. Let

$$\begin{cases} x = \rho\cos\varphi, \\ y = \rho\sin\varphi. \end{cases}$$

Then the Jacobian of transformation

$$J(\rho,\varphi) == \begin{vmatrix} \dfrac{\partial x}{\partial \rho} & \dfrac{\partial x}{\partial \varphi} \\ \dfrac{\partial y}{\partial \rho} & \dfrac{\partial y}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} \cos\varphi & -\rho\sin\varphi \\ \sin\varphi & \rho\cos\varphi \end{vmatrix} = \rho\cos^2\varphi + \rho\sin^2\varphi = \rho.$$

Hence,

$$\iint\limits_{D} f(x,y)dxdy = \iint\limits_{D^*} f(\rho\cos\varphi,\rho\sin\varphi)\rho\, d\rho\, d\varphi. \qquad (4.3)$$

To compute an integral in polar coordinates we have to describe the domain D^* as

$$D^* = \{(\rho, \varphi) \mid \alpha \le \varphi \le \beta,\ \rho_1(\varphi) \le \rho \le \rho_2(\varphi)\}.$$

Then the first integration is performed with respect to φ and the second one to ρ.

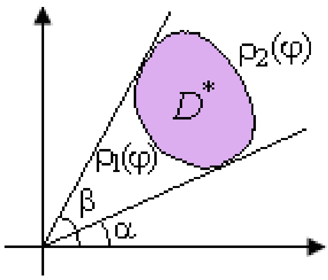


Figure 30.

Hence,

$$\iint_D f(x,y)dxdy = \int_\alpha^\beta d\varphi \int_{\rho_1(\varphi)}^{\rho_2(\varphi)} f(\rho\cos\varphi, \rho\sin\varphi)\rho\, d\rho. \qquad (4.4)$$

Note. Do not forget to multiply by the Jacobian ($|J| = \rho$) and to convert the Cartesian coordinates in the function $f(x,y)$ over to polar coordinates.

Example. Calculate the integral $\iint_D \frac{dxdy}{\sqrt{x^2+y^2}}$, where domain D is bounded by circles $x^2+y^2=1$, $x^2+y^2=4$, lying in the first quarter.

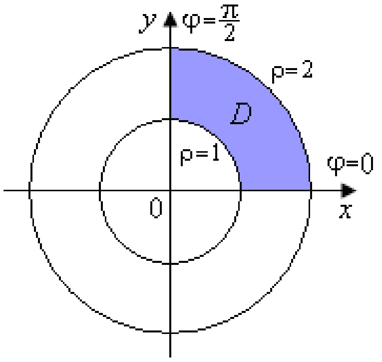


Figure 31.

The domain is a region between two circles (Fig. 31). It would be complicated to compute this integral directly.

Let us make a transformation to polar coordinates: $x = \rho\cos\varphi$, $y = \rho\sin\varphi$. Therefore

$$x^2+y^2 = \rho^2\cos^2\varphi + \rho^2\sin^2\varphi = \rho^2,$$

and

$$1 \le \rho \le 2,\quad 0 \le \varphi \le \frac{\pi}{2}.$$

Finally,

$$\iint_D \frac{dxdy}{\sqrt{x^2+y^2}} = \int_0^{\frac{\pi}{2}} d\varphi \int_1^2 \frac{1}{\sqrt{\rho^2}}\rho d\rho = \int_0^{\frac{\pi}{2}} d\varphi \int_1^2 d\rho = \frac{\pi}{2}(2-1) = \frac{\pi}{2}.$$

4.4 Application of a Double Integral

In this part we find out how we can use double integrals for solving some problems of geometry and mechanics.

I. ***The Area of a Region***

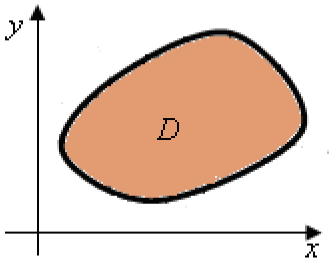


Figure 32.

Consider the region D in the 2D-space (Fig.32). Let us divide the region D into n parts: D_1, D_2, $\ldots D_n$; and denote by ΔS_1, ΔS_2, $\ldots \Delta S_n$ the areas of subdomains.

The area of D is approximately

$$S \approx \sum_{i=1}^{n} \Delta S_i$$

Therefore, it leads to the formula ***the area of a region***

$$S = \iint_D dxdy. \tag{4.5}$$

Example. Find the area of the region D bounded by curves $y = x - 4$, $y^2 = 2x$.

Let us find the points of intersection of the curves by solving equation $y + 4 = 0{,}5\,y^2$. Therefore $(2;-2)$ and $(8;4)$. Plot the graphs and obtain the region D (Fig. 33).

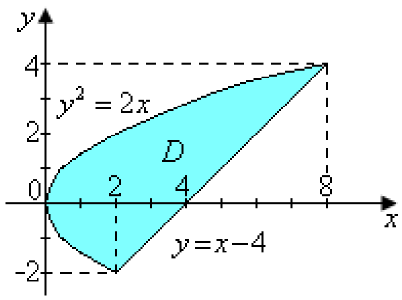


Figure 33.

Describe the region as

$$D = \{(x, y) \mid -2 \le y \le 4,\ 0{,}5y^2 \le x \le y + 4\}.$$

According to formula (4.5)

$$S = \iint_D dxdy = \int_{-2}^{4} dy \int_{0{,}5y^2}^{y+4} dx = \int_{-2}^{4} dy(x)\Big|_{0{,}5y^2}^{y+4} = \int_{-2}^{4} \left(y + 4 - 0{,}5y^2\right) dy =$$

$$= \left(\frac{y^2}{2} + 4y - \frac{y^3}{6}\right)\Bigg|_{-2}^{4} = 18 \text{ (units of area).}$$

II. ***The Volume of the Solid***

Consider the solid $V \in \mathbb{R}^3$ such as its top base is defined by the continuous positive function $z = f(x, y)$, bottom base is a closed region D in xy-plane and side of solid whose directrix is the boundary of D and generators are parallel to z-axis (Fig. 34).

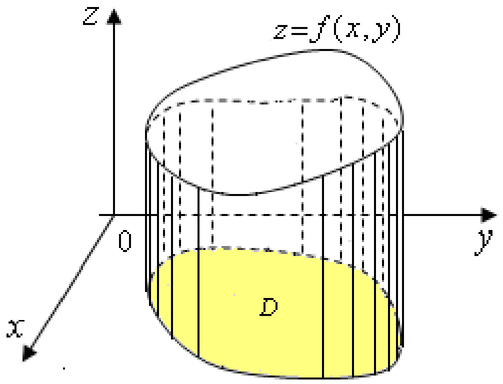


Figure 34.

Previously, in 4.1 I. we obtain the formula of volume of a solid

$$V = \iint_D f(x, y)dxdy. \tag{4.6}$$

Example. Find the volume of solid bounded by surfaces

$$z = x^2 + y^2, \quad x + y = 1, \quad x = 0, \quad y = 0, \quad z = 0.$$

Solid is bounded by coordinate planes, plane $x + y = 1$ (parallel to z-axis) and paraboloid $z = x^2 + y^2$ (Fig. 35).

According to formula (4.6),

$$V = \iint_D (x^2 + y^2)dxdy .$$

Here D is a triangle in xy-plane and

$$D = \{(x, y) \,|\, 0 \le x \le 1, \quad 0 \le y \le 1 - x\}.$$

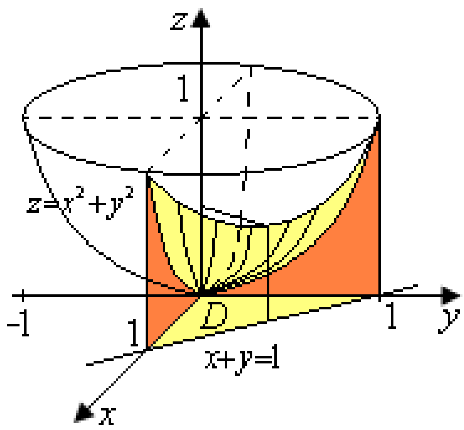


Figure 35.

Hence,

$$V = \iint_D (x^2 + y^2)dxdy = \int_0^1 dx \int_0^{1-x} (x^2 + y^2)dy = \int_0^1 dx \left(x^2 y + \frac{y^3}{3} \right)\Bigg|_0^{1-x} =$$

$$= \int_0^1 \left(x^2 - x^3 + \frac{(1-x)^3}{3} \right)dx = \left(\frac{x^3}{3} - \frac{x^4}{4} - \frac{(1-x)^4}{4} \right)\Bigg|_0^1 = \frac{1}{6} \text{ (units of volume).}$$

III. *The Area of a Surface*

Consider the surface defined by the continuous and differentiable function $z = f(x, y)$. Let us evaluate the area of the part of this surface bounded by a closed curve L. The projection of L on the xy-plane is L_{xy} and the region D_{xy} is the domain on the xy-plane bounded by the curve L_{xy} (Fig. 36).

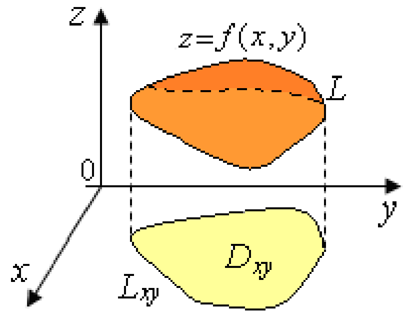


Figure 36.

The formula used to compute the area of a surface is

$$P = \iint_{D_{xy}} \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dxdy\,. \tag{4.7}$$

If the equation of the surface is given in the form $x = g(y, z)$ or in the form $y = h(x, z)$, then the corresponding formulas for calculating the surface area are of the form

$$P = \iint_{D_{yz}} \sqrt{1 + \left(\frac{\partial g}{\partial y}\right)^2 + \left(\frac{\partial g}{\partial z}\right)^2}\, dydz\,, \tag{4.8}$$

$$P = \iint_{D_{xz}} \sqrt{1 + \left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial z}\right)^2}\, dxdz\,, \tag{4.9}$$

where D_{yz} and D_{xz} are the domains in the yz-plane and the xz-plane in which the given surface is projected.

Example. Compute the surface area of the part of hyperbolic paraboloid $z = 1 + (x^2 - y^2)$ that lies in the cylinder given by $x^2 + y^2 = 1$ (Fig. 37).

For calculating the area of a surface let us apply the formula (4.7).

The partial derivatives are

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(1 + (x^2 - y^2)) = 2x\,,$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(1 + (x^2 - y^2)) = -2y\,.$$

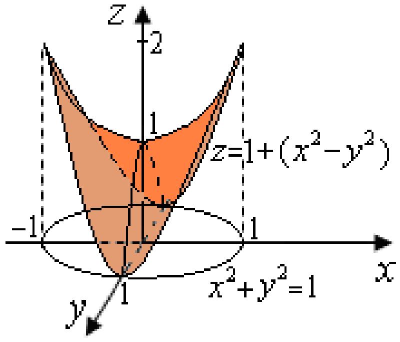


Figure 37.

$$P=\iint\limits_{D_{xy}}\sqrt{1+(2x)^{2}+(-2y)^{2}}dxdy=\iint\limits_{D_{xy}}\sqrt{1+4(x^2+y^2)}dxdy\ ,$$

where D_{xy} the disk of unit radius centered at the origin. It makes sense to do this integral in polar coordinates $D_{xy}=\{(\rho,\varphi)\mid\ 0\le\rho\le1,\ \ 0\le\varphi\le2\pi\}$.

Hence,

$$P=\iint\limits_{D_{xy}}\sqrt{1+4(x^2+y^2)}dxdy=\int_0^{2\pi}d\varphi\int_0^1\rho\sqrt{1+4\rho^2}d\rho=\frac{1}{8}\int_0^{2\pi}d\varphi\int_0^1\sqrt{1+4\rho^2}d(1+4\rho^2)=$$

$$=\frac{1}{8}\int_0^{2\pi}d\varphi\frac{2\sqrt{(1+4\rho^2)^3}}{3}\Bigg|_0^1=\frac{1}{12}(5\sqrt{5}-1)\int_0^{2\pi}d\varphi=\frac{\pi}{6}(5\sqrt{5}-1)\ \ \text{(units of area).}$$

IV. ***The Mass of a Thin Lamina***

Consider a flat, thin object (lamina) of the region D in xy-plane, which is a solid of unit height and whose base is the region D. Let the density varies across the lamina. That is the lamina has a mass-density of $\gamma=\gamma(x,y)$, measured in units of mass per unit area.

Divide the region D into n subdomains: $D_1,\ D_2,\ \ldots D_n$. Denote by $\Delta S_1,\ \Delta S_2,\ \ldots\Delta S_n$ the areas of subdomains. Choose in each D_i a point $P_i(x_i,y_i)$ and calculate a mass-density $\gamma(x_i,y_i)$ at this point.

The mass m of the lamina is approximately the sum of the masses of the subdomains

$$m\approx\sum_{i=1}^{n}\gamma(x_i,y_i)\Delta S_i\ .$$

This is the integral sum of the function $\gamma(x,y)$ in D.

Thus, the total mass over the region D of the mass density $\gamma=\gamma(x,y)$ is

$$m=\iint\limits_{D}\gamma(x,y)dxdy. \tag{4.10}$$

Example. Evaluate the mass of the unite square if a mass-density $\gamma=3x+y$.

According to the formula (4.10) where $D=\{(x,y)\mid\ 0\le x\le1,\ \ 0\le y\le1\}$

$$m=\iint\limits_{D}(3x+y)dxdy=\int_0^1dx\int_0^1(3x+y)dy=\int_0^1dx(3xy+0{,}5y^2)\Big|_0^1=$$

$$=\int_0^1(3x+0{,}5)dx=(1{,}5x^2+0{,}5x)\Big|_0^1=2\ \ \text{(units of mass).}$$

Note. If $\gamma(x,y)=1$ for all points of region D, then the lamina is called homogeneous and its mass is $m=\iint\limits_D dxdy$. That means the mass is the area of region D.

V. ***The Center of Mass of a Thin Lamina***

From physics it is well-known that the center of mass of an object is a point at which the object will balance perfectly. The coordinates of the center of mass of a system of material points $P_1(x_1,y_1)$, $P_2(x_2,y_2)$, … $P_n(x_n,y_n)$ with masses m_1, m_2, … m_n are defined by

$$x_c=\frac{m_1x_1+m_2x_2+\ldots+m_nx_n}{m_1+m_2+\ldots+m_n},\quad y_c=\frac{m_1y_1+m_2y_2+\ldots+m_ny_n}{m_1+m_2+\ldots+m_n}.$$

Let us use this fact to determine the center of mass of the lamina D with mass density $\gamma=\gamma(x,y)$.

Divide the lamina into n subdomains: D_1, D_2, $\ldots D_n$. Denote by ΔS_1, ΔS_2, $\ldots\Delta S_n$ the areas of subdomains. Suppose that the entire mass of D_i is concentrated in some point $P_i(x_i,y_i)$. Then the region D may be regarded as a system of material points and the center of mass of the lamina is approximately

$$x_c=\frac{\sum\limits_{i=1}^{n}x_i\gamma(x_i,y_i)\Delta S_i}{\sum\limits_{i=1}^{n}\gamma(x_i,y_i)\Delta S_i},\quad y_c=\frac{\sum\limits_{i=1}^{n}y_i\gamma(x_i,y_i)\Delta S_i}{\sum\limits_{i=1}^{n}\gamma(x_i,y_i)\Delta S_i}.$$

Passing to the limit as $\max\limits_i diamD_i\to 0$, we obtain exact formulas for the center of mass of lamina D with mass density $\gamma=\gamma(x,y)$

$$x_c=\frac{\iint\limits_D x\gamma(x,y)dxdy}{\iint\limits_D\gamma(x,y)dxdy}=\frac{M_y}{m},\qquad y_c=\frac{\iint\limits_D y\gamma(x,y)dxdy}{\iint\limits_D\gamma(x,y)dxdy}=\frac{M_x}{m}. \tag{4.11}$$

The expressions

$$M_x=\iint\limits_D y\gamma(x,y)dxdy,\qquad M_y=\iint\limits_D x\gamma(x,y)dxdy. \tag{4.12}$$

are called static moments of the thin lamina D with mass density $\gamma=\gamma(x,y)$ relative to the x-axis and y-axis.

Example. Determine the center of mass of the unite square and a mass-density $\gamma = 3x + y$.

Previously the mass of region has been found: $m = 2$ (units of mass).

By formulas (4.11) we have

$$x_c = \frac{\iint\limits_D x(3x+y)dxdy}{m} = \frac{\int\limits_0^1 dx\int\limits_0^1 x(3x+y)dy}{2} = \frac{1}{2}\int\limits_0^1 dx(3x^2y + 0{,}5xy^2)\Big|_0^1 = \frac{1}{2}\int\limits_0^1 (3x^2 + 0{,}5x)dx =$$

$$= \frac{1}{2}\int\limits_0^1 (3x^2 + 0{,}5x)dx = \frac{1}{2}(x^3 + 0{,}25x^2)\Big|_0^1 = \frac{5}{8},$$

$$y_c = \frac{\iint\limits_D y(3x+y)dxdy}{m} = \frac{\int\limits_0^1 dx\int\limits_0^1 y(3x+y)dy}{2} = \frac{1}{2}\int\limits_0^1 dx\left(\frac{3xy^2}{2} + \frac{y^3}{3}\right)\Bigg|_0^1 = \frac{1}{2}\int\limits_0^1 \left(\frac{3x}{2} + \frac{1}{3}\right)dx =$$

$$= \frac{1}{2}\left(\frac{3x^2}{4} + \frac{x}{3}\right)\Bigg|_0^1 = \frac{13}{24}.$$

Hence, the center of mass is

$$(x_c, y_c) = \left(\frac{5}{8}, \frac{13}{24}\right).$$

Note. If the lamina is homogeneous then the center of mass is

$$x_c = \frac{1}{S}\iint\limits_D xdxdy, \qquad y_c = \frac{1}{S}\iint\limits_D ydxdy, \tag{4.13}$$

where S is an area of the region D.

VI. *The Moments of Inertia of a Thin Lamina (Second Area Moments)*

Consider the thin lamina D with mass density $\gamma = \gamma(x, y)$.

The moment of inertia of the lamina relative to the origin:

$$I_0 = \iint\limits_D (x^2 + y^2)\gamma(x, y)dxdy, \tag{4.14}$$

The moment of inertia of the lamina relative to the x-axis and y-axis:

$$I_x = \iint\limits_D y^2\gamma(x, y)dxdy, \qquad I_y = \iint\limits_D x^2\gamma(x, y)dxdy. \tag{4.15}$$

5. Triple Integral

5.1 The Concept of a Triple Integral

Triple integral *for* a function $f = f(x, y, z)$ over a region V in 3D-space can be defined in the same manner as double integral.

Consider a closed solid V in 3D-space and function $f = f(x, y, z)$ that is continuous in V and on its boundary.

Divide the region V into n subdomains: V_1, V_2, $\dots V_n$ and denote by Δv_1, Δv_2, $\dots \Delta v_n$ the volumes of subdomains. Choose an arbitrary point $P_i(x_i, y_i, z_i)$ in each V_i and evaluate $f(x_i, y_i, z_i)$.

Form an integral sum $\sum\limits_{i=1}^{n} f(x_i, y_i, z_i)\Delta v_i$. As $n \to \infty$ then subdomains V_i become smaller and smaller and $\max\limits_{i} diamV_i \to 0$.

Definition. If for continuous in V function $f(x, y, z)$, for any partition of the solid V such that $\max\limits_{i} diamV_i \to 0$ and for any choice of points P_i it exists the limit of integral sum, then that limit $\lim\limits_{\max\limits_{i} diamV_i \to 0} \sum\limits_{i=1}^{n} f(x_i, y_i, z_i)\Delta v_i$ is called ***the triple integral or an integral of a function of three variables over*** V and denoted by

$$\iiint\limits_V f(x, y, z)dv \quad \text{or} \quad \iiint\limits_V f(x, y, z)dxdydz.$$

Properties of triple integrals:

1. Let A, B be an arbitrary real numbers and $f(x, y, z)$, $g(x, y, z)$ be continuous in V functions. Then

$$\iiint\limits_V (Af(x, y, z) \pm Bg(x, y, z))dv = A\iiint\limits_V f(x, y, z)dv \pm B\iiint\limits_V g(x, y, z)dv.$$

2. If a domain V is divided into two domains V_1 and V_2 without common interior points, and a function $f(x, y, z)$ be continuous in all parts of V, then

$$\iiint\limits_V f(x, y, z)dv = \iiint\limits_{V_1} f(x, y, z)dv + \iiint\limits_{V_2} f(x, y, z)dv.$$

5.2 Calculating Triple Integral in Cartesian Coordinates

Suppose that three-dimensional region V bounded by a closed surface S is:

1. every straight line parallel to z-axis cuts the surface S no more than at two points;

2. the entire region V is projected on the xy-plane into regular two-dimensional region D.

Such a domain V is called *regular* three-dimensional region.

Evaluating the triple integrals is similar to computing double integrals. It should be rewrite as a threefold iterated integral.

Let us look through some cases.

Case I. Suppose that V is the rectangular cuboid (Fig. 38)

$$V = \{(x, y, z) \mid a \le x \le b,\ c \le y \le d,\ k \le z \le l\}.$$

Then

$$\iiint_V f(x, y, z)dxdydz = \int_a^b dx \int_c^d dy \int_k^l f(x, y, z)dz.$$

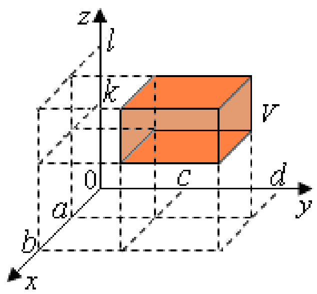


Figure 38.

In this case the integral is independent of the order of integration. That means that

$$\iiint_V f(x,y,z)dv = \int_a^b dx \int_c^d dy \int_k^l f(x,y,z)dz = \int_c^d dy \int_a^b dx \int_k^l f(x,y,z)dz = \int_a^b dx \int_k^l dz \int_c^d f(x,y,z)dy =$$

$$= \int_k^l dz \int_a^b dx \int_c^d f(x,y,z)dy = \int_c^d dy \int_k^l dz \int_a^b f(x,y,z)dx = \int_k^l dz \int_c^d dy \int_a^b f(x,y,z)dx. \qquad (5.1)$$

Case II. Let the solid V be bounded above by the surface $z = z_2(x, y)$ and bounded below by the surface $z = z_1(x, y)$. Suppose that the domain D is the projection of the solid onto the xy-plane (Fig. 39).

Therefore, we can describe the solid as

$$V = \{(x, y, z) \mid (x, y) \in D,\ \ z_1(x, y) \le z \le z_2(x, y)\}.$$

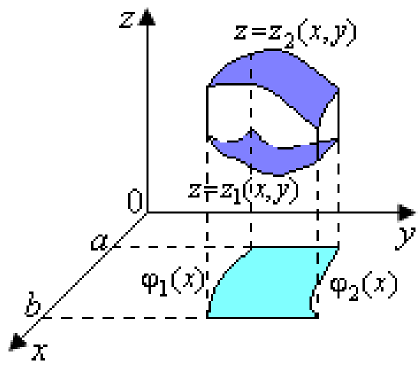


Figure 39.

Hence, the triple integral of the function $f(x,y,z)$ over V is defined as

$$\iiint\limits_V f(x,y,z)dxdydz = \iint\limits_D dxdy \int\limits_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)dz .$$

The inner integral $\int\limits_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)dz = F(x,y)$ is with respect to z. The result is a function of two arguments x and y. As a result, we have

$$\iint\limits_D F(x,y)dxdy .$$

It is a double integral over the domain D in the xy-plane.

If $D = \{(x,y) \mid a \le x \le b,\ \varphi_1(x) \le y \le \varphi_2(x)\}$, then

$$\iiint\limits_V f(x,y,z)dxdydz = \int\limits_a^b dx \int\limits_{\varphi_1(x)}^{\varphi_2(x)} F(x,y)dy .$$

That is if the solid could be described as follows

$$V = \{(x,y,z) \mid a \le x \le b,\ \varphi_1(x) \le y \le \varphi_2(x),\ \ z_1(x,y) \le z \le z_2(x,y)\},$$

then

$$\iiint\limits_V f(x,y,z)dxdydz = \int\limits_a^b dx \int\limits_{\varphi_1(x)}^{\varphi_2(x)} dy \int\limits_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)dz . \tag{5.2}$$

If the solid could be described as

$$V = \{(x,y,z) \mid c \le y \le d,\ \psi_1(y) \le x \le \psi_2(y),\ \ z_1(x,y) \le z \le z_2(x,y)\},$$

then

$$\iiint\limits_V f(x,y,z)dxdydz = \int\limits_c^d dy \int\limits_{\psi_1(y)}^{\psi_2(y)} dx \int\limits_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)dz . \tag{5.3}$$

If the solid V is defined such that

$$V = \{(x,y,z) : (x,z) \in D_{xz},\ \ y_1(x,z) \le y \le y_2(x,z)\},$$

where D_{xz} is a projection of solid onto the xz-plane, then

$$\iiint\limits_V f(x,y,z)dxdydz = \iint\limits_{D_{xz}} dxdz \int\limits_{y_1(x,z)}^{y_2(x,z)} f(x,y,z)dy . \tag{5.4}$$

If the solid V is defined such that

$$V = \{(x,y,z) : (y,z) \in D_{yz},\ \ x_1(y,z) \le x \le x_2(y,z)\},$$

where D_{yz} is a projection of solid onto the yz-plane, then

$$\iiint_V f(x,y,z)dxdydz = \iint_{D_{yz}} dydz \int_{x_1(y,z)}^{x_2(y,z)} f(x,y,z)dx\,. \qquad (5.5)$$

Examples.

1. Calculate the triple integral $\iiint_V (x+y+z)dxdydz$ over the region bounded by planes $x=1,\quad y=1,\quad z=1,\quad x=0,\quad y=0,\quad z=0.$

The solid V is an unit cube. Therefore, let us use formula (5.1).

$$\iiint_V (x+y+z)dv = \int_0^1 dx\int_0^1 dy\int_0^1 (x+y+z)dz = \int_0^1 dx\int_0^1 dy\left(xz+yz+\frac{z^2}{2}\right)\Bigg|_0^1 =$$

$$= \int_0^1 dx\int_0^1\left(x+y+\frac{1}{2}\right)dy = \int_0^1 dx\left(xy+\frac{y^2}{2}+\frac{y}{2}\right)\Bigg|_0^1 = \int_0^1 (1+x)dx = \frac{3}{2}.$$

2. Evaluate the triple integral $\iiint_V zdxdydz$ over the domain bounded by planes $x+y+z=1,\quad x=0,\quad y=0,\quad z=0.$

The solid V is a pyramid and its projection onto the xy-plane is a triangle

$$D=\{(x,y): 0\le x\le 1,\ 0\le y\le 1-x\}.$$

The solid V is bounded above by the plane $z=1-x-y$ and below by the surface $z=0$. That is

$$0\le z\le 1-x-y.$$

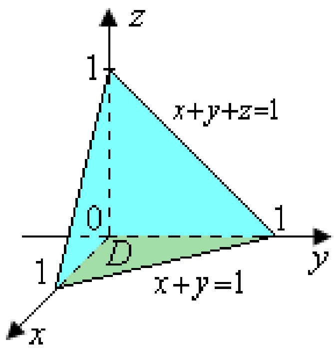


Figure 40.

According to formula (5.2) we have

$$\iiint_V zdv = \int_0^1 dx\int_0^{1-x} dy\int_0^{1-x-y} zdz = \frac{1}{2}\int_0^1 dx\int_0^{1-x}(1-x-y)^2 dy =$$

$$= -\frac{1}{6}\int_0^1 dx(1-x-y)^3\Big|_0^{1-x} = \frac{1}{6}\int_0^1 (1-x)^3 dx = -\frac{(1-x)^4}{24}\Bigg|_0^1 = \frac{1}{24}.$$

5.3 Calculating Triple Integral in Cylindrical and Spherical Coordinates

I. *Change of Variables in a Triple Integral (General Case)*

Consider the triple integral $\iiint\limits_V f(x,y,z)dxdydz$ over the domain V. Let us change of variables in the integral by setting

$$\begin{cases} x = x(u,v,w), \\ y = y(u,v,w), \\ z = z(u,v,w), \end{cases} \qquad (5.6)$$

where functions $x(u,v,w)$, $y(u,v,w)$, and $y(u,v,w)$ are single-valuated, continuous and have continuous partial derivatives. In this case we have a one-to-one correspondence between the points of the domains V in xyz-space and V^* in the uvw-space.

Calculating of the triple integral over a domain V reduces to the evaluation of a triple integral over a domain V^* by the formula

$$\iiint\limits_V f(x,y)dxdydz = \iiint\limits_{V^*} f(x(u,v,w), y(u,v,w), z(u,v,w))\left|J(u,v,w)\right|dudvdw, \qquad (5.7)$$

where $J(u,v,w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} \neq 0$ is called the *Jacobian* or the *functional determinant of transformation*.

II. *The Triple Integral in Cylindrical Coordinates*

In cylindrical coordinates, the position of a point $M(x,y,z)$ in 3D-space determined by three numbers ρ, φ and z, where ρ and φ are polar coordinates of the projection of the point M on the xy-plane and z is the z-coordinates of M (Fig. 41).

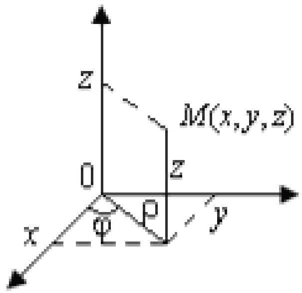


Figure 41.

In this case $M(\rho,\varphi,z)$.

The relationship between rectangular coordinates and cylindrical coordinates is given by formulas

$$\begin{cases} x = \rho\cos\varphi, \\ y = \rho\sin\varphi, \\ z = z, \end{cases} \qquad (5.8)$$

where $0 \le \rho \le +\infty,\ 0 \le \varphi \le 2\pi,\ -\infty \le z \le +\infty$.

Let us calculate the Jacobian of this transformation

$$J(\rho,\varphi,z) = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \varphi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \varphi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \varphi} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\varphi & -\rho\sin\varphi & 0 \\ \sin\varphi & \rho\cos\varphi & 0 \\ 0 & 0 & 1 \end{vmatrix} = \rho \ne 0.$$

Applying general formula (5.7) in the case of cylindrical coordinates we obtain

$$\iiint\limits_V f(x,y,z)dxdydz = \iiint\limits_{V^*} f(\rho\cos\varphi, \rho\sin\varphi, z)\rho\, d\rho\, d\varphi\, dz. \qquad (5.9)$$

Example. Evaluate the integral $\iiint\limits_V \sqrt{x^2+y^2}\,dxdydz$, if solid V is bounded by cylinder $x^2+y^2=4$, paraboloid $z=2+x^2+y^2$ and plane $z=1$ (Fig. 42).

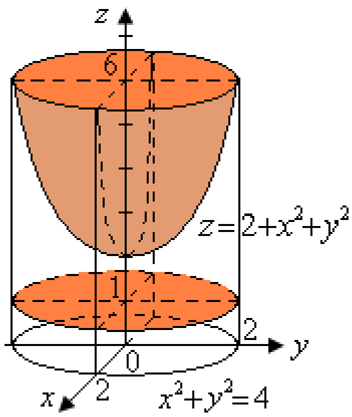


Figure 42.

The projection of solid is a circle of radius 2 centered in origin. To describe such domain, it is better to use polar coordinates. That is why we use cylindrical coordinates for description whole solid.

Since, in cylindrical coordinates, equation of cylinder $x^2+y^2=4$ is $\rho=2$ and equation of paraboloid $z=2+x^2+y^2$ is $z=2+\rho^2$, we can described the solid as follows

$$0 \le \varphi \le 2\pi,\quad 0 \le \rho \le 2,\quad 1 \le z \le 2+\rho^2.$$

Hence,

$$\iiint\limits_V \sqrt{x^2+y^2}\,dxdydz = \int_0^{2\pi} d\varphi \int_0^2 d\rho \int_1^{2+\rho^2} \rho^2 dz = \int_0^{2\pi} d\varphi \int_0^2 (\rho^2+\rho^4)\,d\rho = 2\pi\left(\frac{\rho^3}{3}+\frac{\rho^5}{5}\right)\Bigg|_0^2 = \frac{272}{15}\pi.$$

III. *The Triple Integral in Spherical Coordinates*

In spherical coordinates, the position of a point $M(x,y,z)$ determined by three numbers ρ, φ and θ, where ρ is the distance of the point from the origin, φ is a polar angle of the projection of the point M on the xy-plane and θ is the angle between the radius vector of M and the z-axis (Fig. 43).

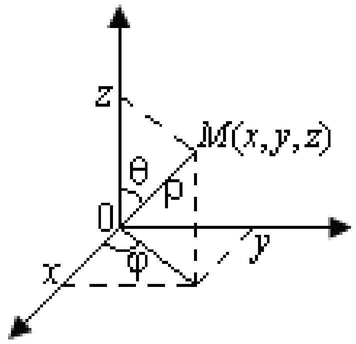


Figure 43.

In this case $M(\rho,\varphi,\theta)$.

The relationship between rectangular coordinates and spherical coordinates is given by formulas

$$\begin{cases} x = \rho\cos\varphi\sin\theta, \\ y = \rho\sin\varphi\sin\theta, \\ z = \rho\cos\theta, \end{cases} \qquad (5.10)$$

where $0 \le \rho \le +\infty,\ 0 \le \varphi \le 2\pi,\ \ 0 \le \theta \le \pi.$

Let us calculate the Jacobian of this transformation

$$J(\rho,\varphi,\theta) = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \varphi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \varphi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \varphi} & \frac{\partial z}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\varphi\sin\theta & -\rho\sin\varphi\sin\theta & \rho\cos\varphi\cos\theta \\ \sin\varphi\sin\theta & \rho\cos\varphi\sin\theta & \rho\sin\varphi\cos\theta \\ \cos\theta & 0 & -\rho\sin\theta \end{vmatrix} =$$

$$= \cos\theta \begin{vmatrix} -\rho\sin\varphi\sin\theta & \rho\cos\varphi\cos\theta \\ \rho\cos\varphi\sin\theta & \rho\sin\varphi\cos\theta \end{vmatrix} - \rho\sin\theta \begin{vmatrix} \cos\varphi\sin\theta & -\rho\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & \rho\cos\varphi\sin\theta \end{vmatrix} =$$

$$= \rho^2\cos^2\theta\sin\theta \begin{vmatrix} -\sin\varphi & \cos\varphi \\ \cos\varphi & \sin\varphi \end{vmatrix} - \rho^2\sin^3\theta \begin{vmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{vmatrix} =$$

$$= \rho^2\cos^2\theta\sin\theta(-\sin^2\varphi - \cos^2\varphi) - \rho^2\sin^3\theta(\sin^2\varphi + \cos^2\varphi) = -\rho^2\cos^2\theta\sin\theta - \rho^2\sin^3\theta =$$

$$= -\rho^2\sin\theta(\cos^2\theta + \sin^2\theta) = -\rho^2\sin\theta \quad \Rightarrow \quad |J(\rho,\varphi,\theta)| = \rho^2\sin\theta.$$

Applying general formula (5.7) in this case we obtain

$$\iiint\limits_V f(x,y,z)\,dxdydz = \iiint\limits_{V^*} f(\rho\cos\varphi\sin\theta, \rho\sin\varphi\sin\theta, \rho\cos\theta)\rho^2\sin\theta\, d\rho\, d\varphi\, dz. \quad (5.11)$$

Note. $x^2 + y^2 + z^2 = \rho^2\cos^2\varphi\sin^2\theta + \rho^2\sin^2\varphi\sin^2\theta + \rho^2\cos^2\theta =$
$= \rho^2\sin^2\theta(\cos^2\varphi + \rho^2\sin^2\varphi) + \rho^2\cos^2\theta = \rho^2\sin^2\theta + \rho^2\cos^2\theta = \rho^2(\sin^2\theta + \cos^2\theta) = \rho^2$

Example. Evaluate the integral $\iiint\limits_V \sqrt{x^2+y^2+z^2}\,dxdydz$, where V is a part of unit ball lying in the first octant (Fig. 44)

Since, the domain of integration is a part of ball, it is better to use spherical coordinates

$$x=\rho\cos\varphi\sin\theta,\quad y=\rho\sin\varphi\sin\theta,\quad z=\rho\cos\theta.$$

In this case we can described the solid as follows

$$0\le\rho\le 1,\ 0\le\varphi\le\frac{\pi}{2},\quad 0\le\theta\le\frac{\pi}{2}.$$

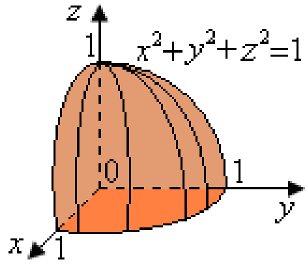


Figure 44.

Therefore,

$$\iiint\limits_V \sqrt{x^2+y^2+z^2}\,dxdydz = \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} d\theta \int_0^1 \rho^3 \sin\theta\, d\rho = \frac{1}{4}\int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} \sin\theta\, d\theta = \frac{1}{4}\int_0^{\frac{\pi}{2}} d\varphi = \frac{\pi}{8}.$$

5.4 Application of the Triple Integral

I. *Volume of a Solid*

If the integrand $f(x,y,z)=1$, then the triple integral over the domain V expresses the volume of the solid V:

$$V=\iiint\limits_V dxdydz\ . \tag{5.12}$$

Example. Find the volume of the solid, bounded by surfaces $y=x,\ x=1,\ y=0,\ z=1,\ z=1+x^2+y^2$ (Fig. 45).

We can describe the solid as follows

$$V=\{(x,y,z): 0\le x\le 1,\ 0\le y\le x,\ 1\le z\le 1+x^2+y^2\}.$$

Hence, according to formula (5.12)

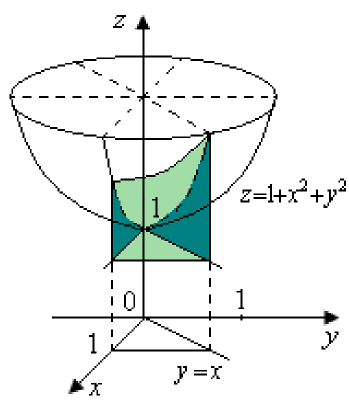


Figure 45.

$$\iiint\limits_V dxdydz = \int_0^1 dx \int_0^x dy \int_1^{1+x^2+y^2} dz = \int_0^1 dx \int_0^x dy\, z\Big|_1^{1+x^2+y^2} =$$

$$=\int_0^1 dx\int_0^x (x^2+y^2)dy = \int_0^1 dx\left(x^2y+\frac{y^3}{3}\right)\Bigg|_0^x = \frac{4}{3}\int_0^1 x^3dx = \frac{4}{3}\left(\frac{x^4}{4}\right)\Bigg|_0^1 = \frac{1}{3}\ \text{(units of volume).}$$

II. *The Mass of a Solid*

Consider the closed solid V and the function $\gamma=\gamma(x,y,z)$. If $\gamma=\gamma(x,y,z)$ continuous positive function, then we can regard it as the density distribution of some substance in the domain V.

The mass of entire substance contained in V is

$$m=\iiint_V \gamma(x,y,z)dxdydz\ . \tag{5.13}$$

Example. Calculate the mass of the solid V, bounded by the surfaces $y=x^2,\quad z=\mathrm{y},\quad y=1,\quad z=0$, if the density is $\gamma=2z+2y$.

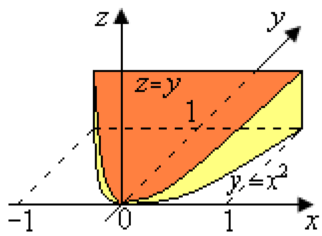


Figure 46.

Let us plot the sketch of a solid (Fig. 46).

The solid is described as follows

$$V=\{(x,y,z):-1\le x\le 1,\ x^2\le y\le 1, 0\le z\le y\}.$$

Hence, by the formula (5.13),

$$\iiint_V(2z+2y)dxdydz=\int_{-1}^{1}dx\int_{x^2}^{1}dy\int_0^y(2z+2y)dz=\int_{-1}^{1}dx\int_{x^2}^{1}dy(z^2+2yz)\Big|_0^y=\int_{-1}^{1}dx\int_{x^2}^{1}3y^2dy=$$

$$=\int_{-1}^{1}dx\,y^3\Big|_{x^2}^{1}=\int_{-1}^{1}(1-x^6)dx=\left(x-\frac{x^7}{7}\right)\Bigg|_{-1}^{1}=\frac{12}{7}\ \text{(units of mass).}$$

III. *The Center of Mass of a Solid*

The coordinates of center of mass of a solid V of the density $\gamma=\gamma(x,y,z)$ are expressed by formulas

$$x_c=\frac{M_{yz}}{m},\quad y_c=\frac{M_{xz}}{m},\quad z_c=\frac{M_{xy}}{m}, \tag{5.14}$$

where m is a mass of solid and M_{xy}, M_{xz}, M_{yz} are static moments of the solid relative to the xy-, xz- and yz-planes respectively.

Static moments could be evaluated by formulas

$$M_{xy}=\iiint_V z\gamma(x,y,z)dxdydz\,,\qquad M_{xz}=\iiint_V y\gamma(x,y,z)dxdydz\,,$$

$$M_{yz}=\iiint_V x\gamma(x,y,z)dxdydz\ . \tag{5.15}$$

Note. If the solid is homogeneous ($\gamma(x,y,z)=1$), then V –his volume, то

$$x_c=\frac{1}{\text{Volume}}\iiint\limits_V xdxdydz\,,\quad y_c=\frac{1}{\text{Volume}}\iiint\limits_V ydxdydz\,,\quad z_c=\frac{1}{\text{Volume}}\iiint\limits_V zdxdydz\,. \qquad (5.16)$$

Example. Determine the coordinates of a center of mass of the upper half of a sphere of radius R with center at the origin, assuming the density is a constant γ_0.

Obviously, by virtue of symmetry of the hemisphere, $x_c=y_c=0$.

The z-coordinate of the center of mass is given by the formula

$$z_c=\frac{\iiint\limits_V z\gamma_0 dxdydz}{\iiint\limits_V \gamma_0 dxdydz}=\frac{\iiint\limits_V zdxdydz}{\iiint\limits_V dxdydz}=\frac{\iiint\limits_V zdxdydz}{\text{Volume of hemisphere}}\,.$$

To calculate the integral in numerator it is better to use spherical coordinates. In this case we can describe this solid as follows

$$0\le\rho\le R,\ 0\le\varphi\le 2\pi,\ \ 0\le\theta\le\pi\,.$$

Hence,

$$z_c=\frac{\iiint\limits_V zdxdydz}{\text{Volume of hemisphere}}=\frac{\int\limits_0^{2\pi}d\varphi\int\limits_0^{\pi}d\theta\int\limits_0^{R}\rho\cos\theta\rho^2\sin\theta d\rho}{\text{Volume of hemisphere}}=\frac{2\pi\frac{R^4}{4}\frac{1}{2}}{\frac{4}{6}\pi R^3}=\frac{\frac{\pi R^4}{4}}{\frac{4}{6}\pi R^3}=\frac{3}{8}R\,.$$

Therefore, the center of mass is $\left(0,\ 0,\ \frac{3}{8}R\right)$.

IV. ***The Moments of Inertia of a Solid***

Consider the solid V with the density $\gamma=\gamma(x,y,z)$.

The moment of inertia of the solid relative to the origin:

$$I_0=\iiint\limits_V(x^2+y^2+z^2)\gamma(x,y,z)dxdydz\,. \qquad (5.17)$$

The moment of inertia of the solid relative to the x-, y- and z-axis respectively:

$$I_x=\iiint\limits_V(y^2+z^2)\gamma(x,y,z)dxdydz\,,\qquad I_y=\iiint\limits_V(x^2+z^2)\gamma(x,y,z)dxdydz\,,$$

$$I_z=\iiint\limits_V(x^2+y^2)\gamma(x,y,z)dxdydz\,. \qquad (5.18)$$

6. Line Integrals with Respect to Arc Length

6.1 The Concept of a Line Integrals with Respect to Arc Length

I. Let us consider the vector $\overrightarrow{OM} = \vec{r}$ in 2D-space whose origin is in the origin of coordinates and whose terminus is in a certain point $M(x, y)$ (Fig. 47). A vector of this kind is called ***a radius vector*.**

It is possible to express this vector in terms of the projection on the coordinate axes:

$$\vec{r} = x\vec{i} + y\vec{j}.$$

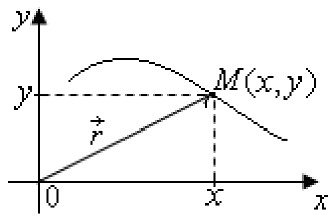


Figure 47.

If the projection of the vector $\vec{r}$ be functions of the parameter t:

$$\begin{cases} x = x(t), \\ y = y(t), \end{cases}$$

then the vector could be rewritten as follows

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j},$$

where t assumes values that lie in the interval $[a, b]$.

As varies, the coordinates vary too, and the point M will trace out a curve on the plane (*godograph*).

Equations

$$\begin{cases} x = x(t), \\ y = y(t), \end{cases}$$

are known as ***the parametric equations*** of the curve, and $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$ is called ***the vector equation*** of the curve.

The vector defined as

$$\frac{d\vec{r}(t)}{dt} = \vec{r}'(t) = x'(t)\vec{i} + y'(t)\vec{j}$$

is called the derivative of vector $\vec{r}(t)$ with respect to the argument t.

The curve $\vec{r}(t)$ is called ***smooth*** on $a \le t \le b$, if is $\vec{r}'(t)$ continuous and $\vec{r}'(t) \ne 0$ for all t. If the curve consists of several smooth parts, the it is called ***piecewise smooth*** curve.

The vector function in 3D-space and its derivative are determined in the same fashion

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k},$$

$$\frac{d\vec{r}(t)}{dt} = \vec{r}\,'(t) = x'(t)\vec{i} + y'(t)\vec{j} + z'(t)\vec{k}.$$

II. Consider the arc of some smooth curve L between points A and B in 2D-space. We called it a *thin wire*.

Assume that $z = f(x, y)$ is a density of wire. This function of two variables is continuous on the region D containing the curve L.

Let us find the mass of wire.

Partition the arc AB into n parts (small arcs) by points $A = A_0,\ A_1,\ \ldots,\ A_n = B$ (Fig.48). Denote the lengths of arcs $\overset{\cup}{A_0A_1},\ \overset{\cup}{A_1A_2}, \ldots \overset{\cup}{A_{n-1}A_n}$ as follows $\Delta l_1,\ \Delta l_2,\ \ldots,\ \Delta l_n$. On each arc $\overset{\cup}{A_{i-1}A_i}$ choose a point $M_i(\xi_i, \eta_i)$ and calculate $f(\xi_i, \eta_i)$.

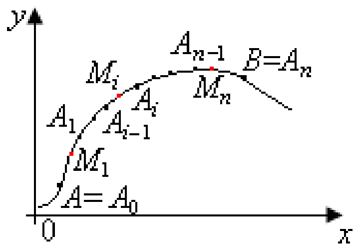


Figure 48.

Write down approximately the mass of wire as a sum

$$m \approx \sum_{i=1}^{n} f(\xi_i, \eta_i)\Delta l_i.$$

As the lengths of arcs approaches zero $\max\limits_{1 \le i \le n} \Delta l_i \to 0$, then the sum gives the mass exactly. That leads us to the concept of the line integral.

Definition. If for continuous on L_{AB} function $f(x, y)$, for any partition of the arc L_{AB} such that $\max\limits_{1 \le i \le n} \Delta l_i \to 0$ and for any choice of points M_i it exists the limit of integral sum $\sum\limits_{i=1}^{n} f(\xi_i, \eta_i)\Delta l_i$, then that limit $\lim\limits_{\substack{\max \Delta l_i \to 0 \\ 1 \le i \le n}} \sum\limits_{i=1}^{n} f(\xi_i, \eta_i)\Delta l_i$ is called ***the line integral with respect to arc length*** and denoted by

$$\int\limits_{L_{AB}} f(x, y)\,d\ell. \tag{6.1}$$

Similarly, the line integral with respect to arc length is determined in 3D-space

$$\int\limits_{L_{AB}} f(x, y, z)\,d\ell. \tag{6.2}$$

Properties of line integrals with respect to arc length:

1. If functions $f(x,y)$ and $g(x,y)$ are continuous on the smooth curve L_{AB}, then for any real numbers C_1 and C_2

$$\int_{L_{AB}}(C_1 f(x,y) \pm C_2 g(x,y))d\ell = C_1 \int_{L_{AB}} f(x,y)d\ell \pm C_2 \int_{L_{AB}} g(x,y)d\ell .$$

2. Let the function $f(x,y)$ is continuous on the curve L_{AB}. Assume, that the curve is consist of two smooth parts: $L_{AB} = L_{AC} \cup L_{CB}$, then

$$\int_{L_{AB}} f(x,y)d\ell = \int_{L_{AC}} f(x,y)d\ell + \int_{L_{CB}} f(x,y)d\ell .$$

3. If the direction, in which the integral $\int_{L_{AB}} f(x,y)d\ell$ is taken along a curve, is reversed, then the value of the line integral doesn't change

$$\int_{L_{AB}} f(x,y)d\ell = \int_{L_{BA}} f(x,y)d\ell .$$

6.2 Calculating Line Integrals with Respect to Arc Length

Let us consider the line integral $\int_{L_{AB}} f(x,y)d\ell$. Assume, that the curve L_{AB} defined in 2D-space as $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$, $t \in [a,b]$.

Since, $\lim\limits_{\substack{\max \Delta l_i \to 0 \\ 1\le i\le n}} \sum_{i=1}^{n} f(\xi_i, \eta_i)\Delta l_i = \int_{L_{AB}} f(x,y)d\ell$, it should be determined the length of each arc: Δl_i. We can find it using Pythagorean theorem

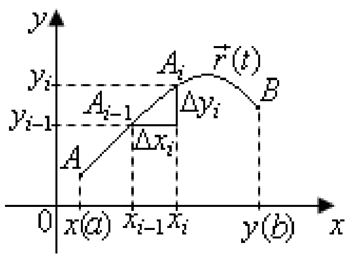


Figure 49.

$$\Delta l_i = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sqrt{\left(\frac{\Delta x_i}{\Delta t_i}\right)^2 + \left(\frac{\Delta y_i}{\Delta t_i}\right)^2}\,\Delta t_i .$$

If $\max\limits_{1\le i\le n} \Delta l_i \to 0$, that is if $\max\limits_{1\le i\le n} \Delta t_i \to 0$, then $d\ell = \sqrt{(x'(t))^2 + (y'(t))^2}\,dt$ and

$$\int_{L_{AB}} f(x,y)d\ell = \int_a^b f(x(t),y(t))\sqrt{(x'(t))^2 + (y'(t))^2}\,dt . \qquad (6.3)$$

Note.

1. If the curve in 2D-space is defined as $y = y(x)$, where $a \le x \le b$, then we can parameterize it as follows

$$\begin{cases} x = x, \\ y = y(x), \end{cases} \text{ or } \vec{r}(x) = x\vec{i} + y(x)\vec{j},\ x \in [a,b].$$

Hence,

$$\int\limits_{L_{AB}} f(x,y)d\ell = \int\limits_a^b f(x,y(x))\sqrt{1+y'^2(x)}dx; \tag{6.4}$$

2. If the curve in 2D-space is defined by its polar equation as $\rho = \rho(\varphi)$, where $\alpha \le \varphi \le \beta$, then we can parameterize it as follows

$$\begin{cases} x = \rho(\varphi)\cos\varphi, \\ y = \rho(\varphi)\sin\varphi, \end{cases} \text{ or } \vec{r}(x) = \rho(\varphi)\cos\varphi\vec{i} + \rho(\varphi)\sin\varphi\vec{j},\ \varphi \in [\alpha,\beta].$$

Thus,

$$\int\limits_{L_{AB}} f(x,y)d\ell = \int\limits_\alpha^\beta f(\rho(\varphi)\cos\varphi, \rho(\varphi)\sin\varphi)\sqrt{\rho^2(\varphi)+\rho'^2(\varphi)}d\varphi. \tag{6.5}$$

3. The curve in 3D-space is defined as

$$\begin{cases} x = x(t), \\ y = y(t), \\ z = z(t), \end{cases} \text{ or } \vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k},\ t \in [a,b].$$

In this case, the line integral with respect to arc length could be calculated by

$$\int\limits_{L_{AB}} f(x,y)d\ell = \int\limits_a^b f(x(t),y(t),z(t))\sqrt{x'^2(t)+y'^2(t)+z'^2(t)}dt. \tag{6.6}$$

Example.

Calculate the line integral $\int\limits_{L_{AB}} xyd\ell$, where L_{AB} is a part of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ lying in the first quarter.

We can describe this part of ellipse in parametric form as

$$\begin{cases} x = a\cos t, \\ y = b\sin t, \end{cases} \quad 0 \le t \le \frac{\pi}{2}.$$

Using formula (6.3) we obtain

$$d\ell = \sqrt{x'^2(t) + y'^2(t)}dt = \sqrt{a^2 \sin^2 t + b^2 \cos^2 t}dt$$

and

$$\int_{L_{AB}} xyd\ell = \int_0^{\frac{\pi}{2}} ab \cos t \sin t \sqrt{a^2 \sin^2 t + b^2 \cos^2 t}dt =$$

$$= \frac{ab}{2(a^2 - b^2)} \int_0^{\frac{\pi}{2}} \sqrt{a^2 \sin^2 t + b^2 \cos^2 t}d(a^2 \sin^2 t + b^2 \cos^2 t) =$$

$$= \frac{ab}{3(a^2 - b^2)} \sqrt{(a^2 \sin^2 t + b^2 \cos^2 t)^3}\Bigg|_0^{\frac{\pi}{2}} = \frac{ab}{3(a^2 - b^2)}(a^3 - b^3) = \frac{ab(a^2 + ab + b^2)}{3(a+b)}.$$

6.3 Application of the Line Integrals with Respect to Arc Length

I. ***The Length of the Arc***

Consider the arc of some smooth curve L between points A and B. The length of the arc L_{AB} is

$$L = \int_{L_{AB}} d\ell . \tag{6.7}$$

Example. Find the length of Cardioid $\rho = a(1 + \cos\varphi)$ (Fig. 50)

The curve is symmetric with respect to coordinate axis. That's why it is enough to find the length of the part lying in the upper part of the plane.

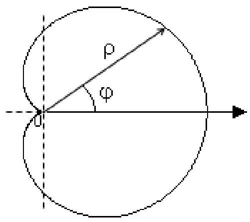


Figure 50.

Since,

$$\rho = a(1 + \cos\varphi),\ 0 \le \varphi \le \pi ,$$

we have

$$d\ell = \sqrt{(a(1 + \cos\varphi))^2 + (-a\sin\varphi)^2}d\varphi = \sqrt{a^2 + 2a^2 \cos\varphi + a^2 \cos^2\varphi + a^2 \sin^2\varphi}d\varphi =$$

$$= \sqrt{2a^2 + 2a^2 \cos\varphi}d\varphi = a\sqrt{2(1 + \cos\varphi)}d\varphi = a\sqrt{4\cos^2\frac{\varphi}{2}}d\varphi = 2a\cos\frac{\varphi}{2}d\varphi .$$

According to formula (6.7)

$$L = \int_{L_{AB}} d\ell = 2\int_0^{\pi} 2a\cos\frac{\varphi}{2}d\varphi = 8a\sin\frac{\varphi}{2}\Bigg|_0^{\pi} = 8a \text{ (units of length).}$$

II. ***The Area of Cylindrical Surface***

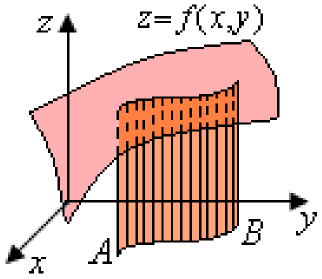


Figure 51.

Consider the cylindrical surface between the curve L between points A and B in the xy-plane and the projection of this curve on the surface $z = f(x,y)$. The area of such surface is determined by formula

$$P = \int_{L_{AB}} f(x,y)d\ell . \tag{6.8}$$

Example. Find the area of the part of the cylindrical surface $x^2 + y^2 = 4$, that is cut by the xy-plane and the surface $z = 2 + \frac{x^2}{2}$.

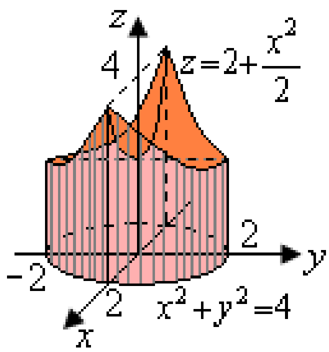


Figure 52.

Let us use the formula (6.8)

$$P = \int_L \left(2 + \frac{x^2}{2}\right) d\ell ,$$

where L is a circle $x^2 + y^2 = 4$. To calculate this integral, it is better to use parametric equations

$$\begin{cases} x = 2\cos t, \\ y = 2\sin t, \end{cases} \quad 0 \le t \le 2\pi .$$

Hence,

$$d\ell = \sqrt{(-2\sin\varphi)^2 + (2\cos\varphi)^2}\, d\varphi = 2d\varphi$$

and

$$P = \int_L \left(2 + \frac{x^2}{2}\right) d\ell = \int_0^{2\pi} (2 + 2\cos^2 t)2dt = \int_0^{2\pi} (6 + 2\cos 2t)dt = 12\pi \text{ (units of area).}$$

III. ***The Mass and the Center of Mass of the Thin Wire***

Consider a thin wire in the shape of the curve L in 2D-space with the given density function $\gamma = \gamma(x,y)$.

Then the mass of thin wire is

$$m = \int_L \gamma(x,y)d\ell . \tag{6.9}$$

The center of mass is

$$x_c = \frac{M_y}{m} = \frac{\int_L x\gamma(x,y)d\ell}{\int_L \gamma(x,y)d\ell}, \qquad y_c = \frac{M_x}{m} = \frac{\int_L y\gamma(x,y)d\ell}{\int_L \gamma(x,y)d\ell}. \tag{6.10}$$

Note. For the thin wire L in 3D-space with the density $\gamma = \gamma(x,y,z)$ we have

$$m = \int_L \gamma(x,y,z)d\ell; \tag{6.11}$$

$$x_c = \frac{1}{m}\int_L x\gamma(x,y,z)d\ell, \quad y_c = \frac{1}{m}\int_L y\gamma(x,y,z)d\ell, \quad z_c = \frac{1}{m}\int_L z\gamma(x,y,z)d\ell. \tag{6.12}$$

Examples.

1. Find the mass of lemniscate of Bernoulli $\rho = a\sqrt{\cos 2\varphi}$ (Fig. 53) lying in the first quarter, if the density is $\gamma = x$.

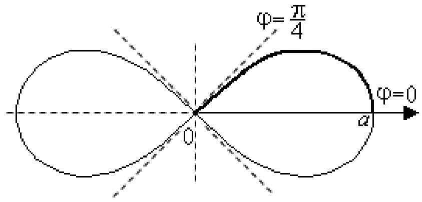


Figure 53.

Since, $\rho = a\sqrt{\cos 2\varphi}$, $0 \le \varphi \le \dfrac{\pi}{4}$, we have

$$d\ell = \sqrt{(a\sqrt{\cos 2\varphi})^2 + \left(\frac{-2a\sin 2\varphi}{2\sqrt{\cos 2\varphi}}\right)^2}d\varphi = \sqrt{a^2\cos 2\varphi + \frac{a^2\sin^2 2\varphi}{\cos 2\varphi}}d\varphi = \frac{a}{\sqrt{\cos 2\varphi}}d\varphi$$

Hence, by the formula (6.10)

$$m = \int_L x d\ell = \int_0^{\frac{\pi}{2}} \rho(\varphi)\cos\varphi \frac{a}{\sqrt{\cos 2\varphi}}d\varphi = \int_0^{\frac{\pi}{2}} a\sqrt{\cos 2\varphi}\cos\varphi \frac{a}{\sqrt{\cos 2\varphi}}d\varphi =$$

$$= a^2\int_0^{\frac{\pi}{2}}\cos\varphi d\varphi = a^2 \sin\varphi\Big|_0^{\frac{\pi}{2}} = a^2 \text{ (units of mass).}$$

2. Evaluate the coordinates of mass of the part of circular helix $x = a\cos t$, $y = a\sin t$, $z = bt$, $0 \le t \le \pi$ (Fig. 54), if the density γ is a constant.

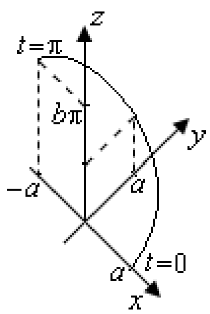


Figure 54.

First, let us find $d\ell$

$$d\ell = \sqrt{x'^2(t) + y'^2(t) + z'^2(t)}dt = \sqrt{a^2\sin^2 t + a^2\cos^2 t + b^2}dt =$$

$$= \sqrt{a^2 + b^2}dt.$$

Next, let us evaluate the mass by formula (6.11)

$$m = \int_0^{\pi} \gamma d\ell = \gamma \int_0^{\pi} \sqrt{a^2 + b^2} dt = \pi\gamma \sqrt{a^2 + b^2} \text{ (units of mass).}$$

Let us find the coordinates of the center of mass by formulas (6.12)

$$x_c = \frac{\gamma}{m} \int_0^{\pi} a \cos t \sqrt{a^2 + b^2} dt = 0, \qquad y_c = \frac{\gamma}{m} \int_0^{\pi} a \sin t \sqrt{a^2 + b^2} dt = \frac{2a}{\pi},$$

$$z_c = \frac{\gamma}{m} \int_0^{\pi} bt \sqrt{a^2 + b^2} dt = \frac{b\pi}{2}.$$

IV. ***Moments of Inertia of the Thin Wire***

Consider a thin wire in the shape of the curve L in 2D-space with the given density function $\gamma = \gamma(x, y)$.

The moment of inertia with respect to origin

$$I_0 = \int_L (x^2 + y^2)\gamma(x, y) d\ell . \qquad (6.13)$$

The moment of inertia with respect to coordinate axes

$$I_x = \int_L y^2 \gamma(x, y) d\ell, \quad I_y = \int_L x^2 \gamma(x, y) d\ell . \qquad (6.14)$$

Note. For the thin wire L in 3D-space with the density $\gamma = \gamma(x, y, z)$ we have

$$I_0 = \int_L (x^2 + y^2 + z^2)\gamma(x, y, z) d\ell , \qquad (6.15)$$

$$I_x = \int_L y^2 \gamma(x, y, z) d\ell, \quad I_y = \int_L x^2 \gamma(x, y, z) d\ell, \quad I_z = \int_L z^2 \gamma(x, y, z) d\ell . \qquad (6.16)$$

7. Surface Integral Over the Surface

7.1 The Concept of a Surface Integral Over the Surface

Consider the function $z = z(x, y)$ that defines the closed part S of surface in 3D-space. Let D_{xy} be the projection of the surface S onto the xy-plane such that the area of the region exist (*regular projection*).

Suppose that continuous function $f(x, y, z)$ is a density per unit area of the surface.

Let us find the mass of the surface S.

Partition the surface S into n parts (subsurfaces): S_1, S_2, …, S_n (Fig.55). Denote the areas of parts as follows $\Delta\sigma_1$, $\Delta\sigma_2$, …, $\Delta\sigma_n$. In each subsurface S_i choose a point $M_i(\xi_i, \eta_i, \varsigma_i)$ and evaluate $f(\xi_i, \eta_i, \varsigma_i)$ (mass at the point M_i).

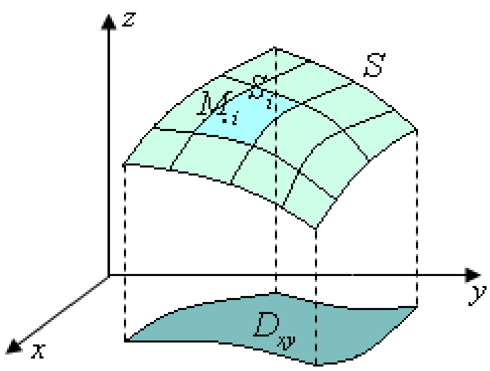


Figure 55.

Write down approximately the mass of surface as a sum

$$m \approx \sum_{i=1}^{n} f(\xi_i, \eta_i, \varsigma_i)\Delta\sigma_i .$$

As the areas of subsurfaces approaches zero $\max\limits_{1\le i\le n} \Delta\sigma_i \to 0$, then the sum gives the mass exactly. That leads us to the concept of the surface integral over the surface.

Definition. If for continuous function $f(x, y, z)$, for any partition of the surface S such that $\max\limits_{1\le i\le n} \Delta\sigma_i \to 0$ and for any choice of points M_i it exists the limit of integral sum $\sum\limits_{i=1}^{n} f(\xi_i, \eta_i, \varsigma_i)\Delta\sigma_i$, then that limit $\lim\limits_{\substack{\max \Delta\sigma_i \to 0 \\ 1\le i\le n}} \sum\limits_{i=1}^{n} f(\xi_i, \eta_i, \varsigma_i)\Delta\sigma_i$ is called ***the surface integral over the surface*** and denoted by

$$\iint_S f(x, y, z)d\sigma . \tag{7.1}$$

Properties of a Surface Integral Over the Surface

1. If functions $f(x,y,z)$ and $g(x,y,z)$ are continuous on the surface S, then for any real numbers C_1 and C_2

$$\iint_S [C_1 f(x,y,z) + C_2 g(x,y,z)]d\sigma = C_1 \iint_S f(x,y,z)d\sigma + C_2 \iint_S g(x,y,z)d\sigma .$$

2. Let the function $f(x,y,z)$ is continuous on the surface S. Assume, that the surface is consist of two parts S_1 and S_2: $S = S_1 \cup S_2$, then

$$\iint_S f(x,y,z)d\sigma = \iint_{S_1} f(x,y,z)d\sigma + \iint_{S_2} f(x,y,z)d\sigma .$$

7.2 Calculating Surface Integral Over the Surface

Consider the surface S defined by equation $z = z(x,y)$. Let D_{xy} be the projection of the surface S onto the xy-plane. Suppose that functions $z(x,y)$, $z'_x(x,y)$, $z'_y(x,y)$ are continuous at points of the region D_{xy}.

Let us calculate the surface integral over the surface

$$\iint_S f(x,y,z)d\sigma .$$

In this case

$$d\sigma = \sqrt{1 + {z'_x}^2(x,y) + {z'_y}^2(x,y)}dxdy$$

and

$$\iint_S f(x,y,z)d\sigma = \iint_{D_{xy}} f(x,y,z(x,y))\sqrt{1 + {z'_x}^2(x,y) + {z'_y}^2(x,y)}dxdy. \tag{7.2}$$

If the surface is defined by $y = y(x,z)$ and its projection onto the xz-plane is D_{xz}. Then

$$\iint_S f(x,y,z)d\sigma = \iint_{D_{xz}} f(x,y(x,z),z)\sqrt{1 + {y'_x}^2(x,z) + {y'_z}^2(x,z)}dxdz , \tag{7.3}$$

where $y(x,z)$, $y'_x(x,z)$, $y'_z(x,z)$ are continuous at points of the region D_{xz}.

For the surface defined by $x = x(y,z)$ we have

$$\iint_S f(x,y,z)d\sigma = \iint_{D_{yz}} f(x(y,z),y,z)\sqrt{1 + {x'_y}^2(y,z) + {x'_z}^2(y,z)}dydz\ , \tag{7.4}$$

where $x(y,z)$, $x'_y(y,z)$, $x'_z(y,z)$ are continuous at points of the region D_{xz} (the projection of S onto the yz-plane).

For the surface S defined parametrically

$$\begin{cases} x = x(u,v), \\ y = y(u,v), \quad (u,v) \in D, \\ z = z(u,v), \end{cases}$$

we have

$$d\sigma = \sqrt{J_1^2(u,v) + J_2^2(u,v) + J_3^2(u,v)}dudv,$$

where

$$J_1(u,v) = \begin{vmatrix} y'_u & z'_u \\ y'_v & z'_v \end{vmatrix} = y'_u z'_v - y'_v z'_u\ ,$$

$$J_2(u,v) = \begin{vmatrix} z'_u & x'_u \\ z'_v & x'_v \end{vmatrix} = z'_u x'_v - z'_v x'_u\ , \tag{7.5}$$

$$J_3(u,v) = \begin{vmatrix} x'_u & y'_u \\ x'_v & y'_v \end{vmatrix} = x'_u y'_v - x'_v y'_u\ .$$

Hence,

$$\iint_S f(x,y,z)d\sigma = \iint_D f(x(u,v),y(u,v),z(u,v))\sqrt{J_1^2(u,v) + J_2^2(u,v) + J_3^2(u,v)}dudv \tag{7.6}$$

Example. Evaluate the surface integral

$$\iint_S (x + 2y - z)d\sigma$$

over the part of plane $x + 2y + 3z = 6$ lying in the first octant.

Rewrite the equation of plane as $z = \dfrac{6 - x - 2y}{3}$ and use the formula (7.2), where D_{xy} is a triangle in xy-plane bounded by $x = 0$, $y = 0$, $x + 2y = 6$ (Fig. 56).

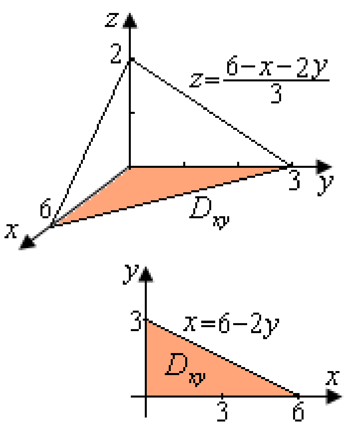


Figure 56.

Since,

$$d\sigma = \sqrt{1 + z_x'^2(x,y) + z_y'^2(x,y)}dxdy = \sqrt{1 + \left(-\frac{1}{3}\right)^2 + \left(-\frac{2}{3}\right)^2}dxdy = \frac{\sqrt{14}}{3}dxdy\ ,$$

we have

$$\iint_S (x + 2y - z)d\sigma = \iint_{D_{xy}} \left(x + 2y - \frac{6 - x - 2y}{3}\right)\frac{\sqrt{14}}{3}dxdy = \frac{\sqrt{14}}{9}\int_0^3 dy \int_0^{6-2y}(4x + 8y - 6)dx =$$

$$= \frac{\sqrt{14}}{9}\int_0^3 dy(2x^2 + 8xy - 6x)\Big|_0^{6-2y} = \frac{\sqrt{14}}{9}\int_0^3 (2(6-2y)^2 + 60y - 16y^2 - 36)dy =$$

$$= \frac{\sqrt{14}}{9}\left(-\frac{(6-2y)^3}{3} + 30y^2 - \frac{16}{3}y^3 - 36y\right)\Bigg|_0^3 = 10\sqrt{14}\ .$$

7.3 Application of the Surface Integral Over the Surface

I. ***The Area of the Surface***

Consider the surface S in 3D-space, then the area of this surface is

$$A_S = \iint_S d\sigma\ . \qquad (7.7)$$

Example. Find the area of a part of cone $z = \sqrt{x^2 + y^2}$, cut off by the cylinder $x^2 + y^2 = 2x$.

Let us use the formula (7.7).

Calculate $d\sigma$:

$$d\sigma = \sqrt{1 + \left(\frac{2x}{2\sqrt{x^2+y^2}}\right)^2 + \left(\frac{2y}{2\sqrt{x^2+y^2}}\right)^2}dxdy = \sqrt{2}dxdy\ .$$

Then

$$A_S = \iint_S d\sigma = \iint_{D_{xy}} \sqrt{2}dxdy\ .$$

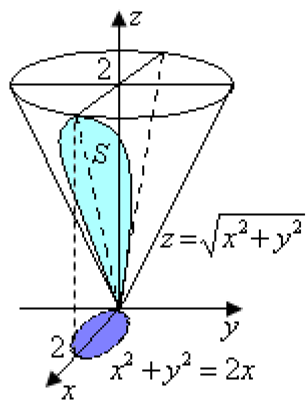


Figure 57.

The projection D_{xy} is a region in xy-plane bounded by the circle $x^2 + y^2 = 2x$. To evaluate the integral, it is better to use polar coordinates.

Thus,

$$D_{xy} = \left\{ (\rho, \varphi) \mid -\frac{\pi}{2} \le \varphi \le \frac{\pi}{2}, \quad 0 \le \rho \le 2\cos\varphi \right\}.$$

Hence,

$$A_S = \iint_{D_{xy}} \sqrt{2}dxdy = 2\int_0^{\frac{\pi}{2}} d\varphi \int_0^{2\cos\varphi} \rho\sqrt{2}d\rho = \sqrt{2}\int_0^{\frac{\pi}{2}} d\varphi \rho^2 \Big|_0^{2\cos\varphi} = 4\sqrt{2}\int_0^{\frac{\pi}{2}} \cos^2\varphi d\varphi =$$

$$= 2\sqrt{2}\int_0^{\frac{\pi}{2}} (1+\cos 2\varphi)d\varphi = 2\sqrt{2}\left(1+\frac{\sin 2\varphi}{2}\right)\Bigg|_0^{\frac{\pi}{2}} = \sqrt{2}\pi \text{ (units of area).}$$

II. ***The Mass of the Surface***

Consider the surface S in 3D-space and $\gamma = \gamma(x, y, z)$ is a density per unit area of the surface. The mass of the surface is

$$m = \iint_S \gamma(x, y, z)d\sigma. \tag{7.8}$$

III. ***The Coordinates of the Center of Mass of the Surface***

The coordinates of the center of mass of the surface S, if $\gamma = \gamma(x, y, z)$ is a density per unit area of the surface, are

$$x_c = \frac{1}{m}\iint_S x\gamma(x, y, z)d\sigma, \quad y_c = \frac{1}{m}\iint_S y\gamma(x, y, z)d\sigma, \quad z_c = \frac{1}{m}\iint_S z\gamma(x, y, z)d\sigma. \tag{7.9}$$

IV. ***Moments of Inertia of the Surface***

Consider the surface S and $\gamma = \gamma(x, y, z)$ is a density per unit area of the surface.

The moment of inertia with respect to origin

$$I_0 = \iint_S (x^2 + y^2 + z^2)\gamma(x, y, z)d\sigma. \tag{7.10}$$

The moment of inertia with respect to coordinate axes

$$I_x = \iint_S (y^2 + z^2)\gamma(x, y, z)d\sigma, \quad I_y = \iint_S (x^2 + z^2)\gamma(x, y, z)d\sigma,$$
$$I_z = \iint_S (x^2 + y^2)\gamma(x, y, z)d\sigma. \tag{7.11}$$

8. Scalar and Vector Fields

Studying physics, we often use two kinds of quantities: scalars and vectors. The important difference between them is that a scalar is characterized only by a magnitude, but a vector has both characteristics: a magnitude and a direction (see Appendix 5).

Examples of scalar quantities are mass, pressure, temperature, electric charge, distance, etc. Examples of vector quantities are velocity, acceleration, magnetic field, etc.

8.1 The Concept of a Scalar Field

Consider the function $u = u(x_1, x_2, \dots x_n)$ defined for each point $(x_1, x_2, \dots x_n)$ of the region $D \subset \mathbb{R}^n$. In this case we say that a *scalar field* is defined in the region D. That means it associates a real number with every position in some region.

For example, if $u = u(x, y)$ denotes the temperature at the point (x, y) of a thin lamina $D \subset \mathbb{R}^2$, then we say that a *scalar field of temperatures* is defined. If $D \subset \mathbb{R}^3$ is filled with a liquid and $u = u(x, y, z)$ denotes pressure then we have an example of a scalar field of pressure. Another example of scalar field is the electrostatic potential $u = u(x, y, z)$.

For better understanding of scalar fields we use *level curves* $u(x, y) = C$ (see 1.1) and *level surfaces* $u(x, y, z) = C$, where $C \in \mathbb{R}$.

8.2 The Concept of a Vector Field

A vector field in 2D-space is a function that assigns to each point $(x, y) \in D \subset \mathbb{R}^2$ a two-dimensional vector $\vec{F}(x, y)$:

$$\vec{F} = \vec{F}(x, y) = \big(P(x, y),\ Q(x, y)\big) = P(x, y)\vec{i} + Q(x, y)\vec{j}\,.$$

The x- and y-components are scalar functions (fields) and the vector field is defined as a vector-function of the location.

In a similar manner, a vector field in 3D-space is defined as a three-dimensional vector $\vec{F}(x, y, z)$:

$$\vec{F} = \vec{F}(x,y,z) = \left(P(x,y,z),\ Q(x,y,z), R(x,y,z)\right) = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}\ .$$

A vector field can be visualized as a collection of arrows with a given magnitude and direction, each attached to a point in the space.

For example, the sketch of vector field $\vec{F} = -x\vec{i} + y\vec{j}$ is on the Figure 58.

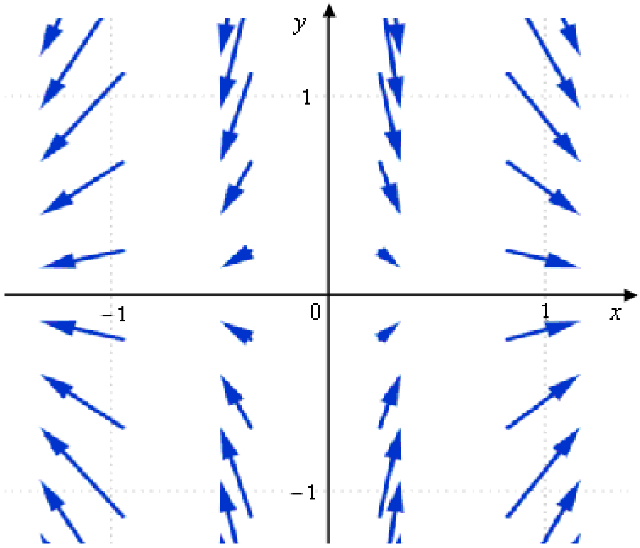


Figure 58.

The sketch of three-dimensional vector field $\vec{F} = z\vec{i} + x\vec{j} + y\vec{k}$ is on the Figure 59.

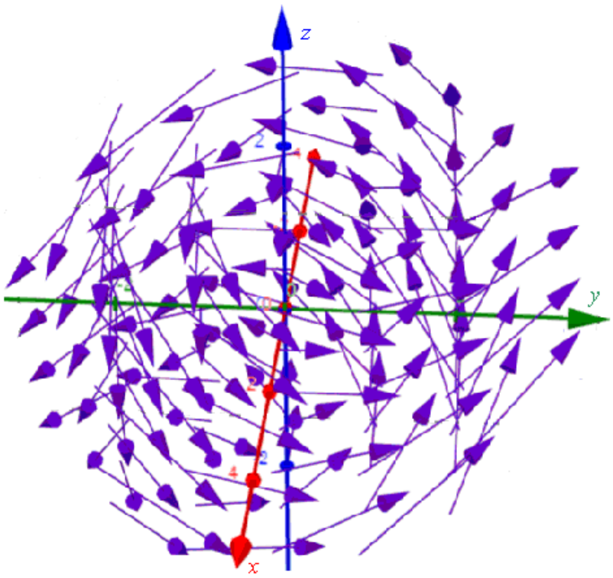


Figure 59.

Since vectors have two characteristics: a magnitude and a direction, then it is naturally use them for modeling physical processes, for example, for researching of a motion of a fluid throughout the space (velocity fields) or of an action of some force (the magnetic, electric or gravitational force) (Fig. 60).

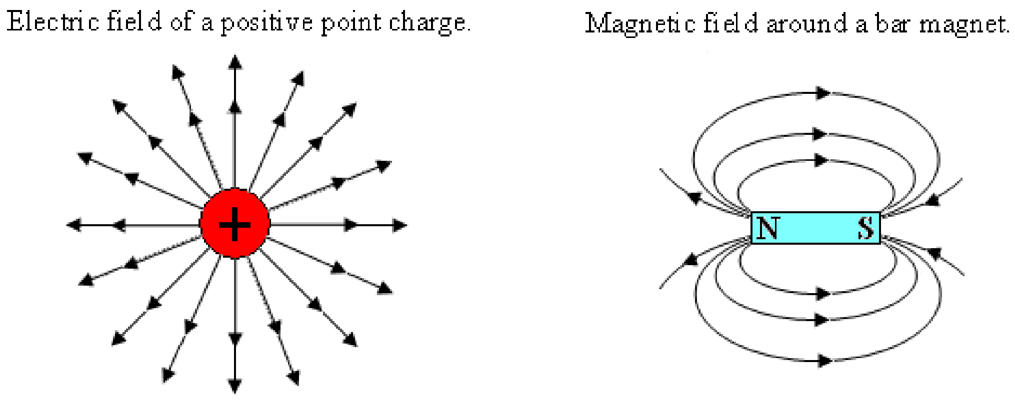


Figure 60.

8.3 The Gradient Vector and Directional Derivative

Let us consider the scalar field $u = u(x, y)$ defined in some region D in 2D space. Assume that the function $u = u(x, y)$ is continuous and has continuous partial derivatives with respect to arguments x and y in this region.

Partial derivative $\frac{\partial u}{\partial x}$ represent the rate of change with respect to variable x when variable y remains unchanged. Similarly, $\frac{\partial u}{\partial y}$ represent the rate of change with respect to y when x is fixed. Let us find the rate of change of functions if both variables x and y change simultaneously. Since, there are a lot of ways of changing variables x and y, we should define the direction we want to find the rate of change.

Suppose that direction is defined by unit vector $\vec{a} = a_x\vec{i} + a_y\vec{j}$, that is $|\vec{a}| = 1$.

Definition. The rate of change of the function $u = u(x, y)$ in the direction of the unit vector $\vec{a} = a_x\vec{i} + a_y\vec{j}$ is called the ***directional derivative*** and is denoted by

$$\frac{\partial u}{\partial \vec{a}} = \lim_{\varepsilon \to 0} \frac{u(x + a_x\varepsilon, y + a_y\varepsilon) - u(x, y)}{\varepsilon}.$$

Let us calculate this limit. Consider the function of one variable

$$f(t) = u(x + a_x t, y + a_y t),$$

where $x,\ a_x,\ y,\ a_y$ are known numbers.

Then, the limit from the definition represent the derivative of $f(t)$ with respect to t at the point $t=0$:

$$\lim_{\varepsilon\to 0}\frac{u(x+a_x\varepsilon, y+a_y\varepsilon)-u(x,y)}{\varepsilon}=\lim_{\varepsilon\to 0}\left.\frac{f(t+\varepsilon)-f(t)}{\varepsilon}\right|_{t=0}=\lim_{\varepsilon\to 0}\frac{f(\varepsilon)-f(0)}{\varepsilon}=f'(0).$$

Since,

$$f'(0)=\left.f'(t)\right|_{z=0}=\left.\left(\frac{\partial u}{\partial(x+a_xt)}\frac{\partial(x+a_xt)}{\partial z}+\frac{\partial u}{\partial(y+a_yt)}\frac{\partial(y+a_yt)}{\partial z}\right)\right|_{t=0}=\frac{\partial u}{\partial x}a_x+\frac{\partial u}{\partial y}a_y,$$

we obtain the formula for calculating the directional derivative

$$\frac{\partial u}{\partial \vec{a}}=\frac{\partial u}{\partial x}a_x+\frac{\partial u}{\partial y}a_y. \tag{8.1}$$

We can rewrite this formula using dot product of two vectors

$$\frac{\partial u}{\partial \vec{a}}=\frac{\partial u}{\partial x}a_x+\frac{\partial u}{\partial y}a_y=\left(\frac{\partial u}{\partial x}\vec{i}+\frac{\partial u}{\partial y}\vec{j}\right)\cdot\left(a_x\vec{i}+a_y\vec{j}\right).$$

Notice that the second vector is $\vec{a}=a_x\vec{i}+a_y\vec{j}$ that gives the direction of change. The first vector $\frac{\partial u}{\partial x}\vec{i}+\frac{\partial u}{\partial y}\vec{j}$ consists of partial derivatives of a function $u=u(x,y)$.

Definition. The vector consisting of partial derivatives is called ***gradient*** or ***gradient vector*** and is denoted by

$$\text{grad } u=\frac{\partial u}{\partial x}\vec{i}+\frac{\partial u}{\partial y}\vec{j} \text{ or } \nabla u=\frac{\partial u}{\partial x}\vec{i}+\frac{\partial u}{\partial y}\vec{j}. \tag{8.2}$$

Gradient vector is **perpendicular** to the level curve $u(x,y)=C$ lying in the xy-plane and passing through the corresponding point.

Hence, the general formula for calculating directional derivative is

$$\frac{\partial u}{\partial \vec{a}}=\text{grad } u\cdot\vec{a}=\nabla u\cdot\vec{a}. \tag{8.3}$$

Since, vector $\vec{a}$ is an unit vector, we can rewrite formula (8.3) as

$$\frac{\partial u}{\partial \vec{a}}=\frac{\nabla u\cdot\vec{a}}{|\vec{a}|}=\text{pr}_{\vec{a}}\text{grad } u,$$

that means, that the derivative of $u=u(x,y)$ along the direction of the vector $\vec{a}$ is equal to the projection of the vector $\text{grad}\ u$ on the vector $\vec{a}$.

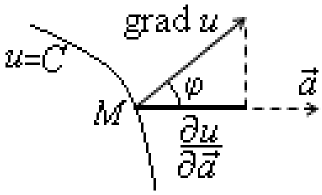


Figure 61.

This rule gives us the relationship between the gradient and the directional derivative at a given point (Fig. 61).

We can rewrite the dot product of vectors in the formula (8.4) as

$$\frac{\partial u}{\partial \vec{a}} = \text{grad}\ u \cdot \vec{a} = |\text{grad}\ u||\vec{a}|\cos\varphi = |\text{grad}\ u|\cos\varphi,$$

where φ is the angle between the gradient and $\vec{a}$.

The directional derivative depends on the angle φ. It takes on its greatest positive value if $\varphi = 0$. In this case

$$\frac{\partial u}{\partial \vec{a}} = |\text{grad}\ u|.$$

Since directional derivative represent the rate of change of function along the direction of vector, the **maximum rate of change** of the scalar field $u = u(x, y)$ is given by $|\text{grad}\ u|$ and will occur in the direction given by $\text{grad}\ u$ (Fig. 62) and

$$v_{\max} = |\text{grad}\ u|.$$

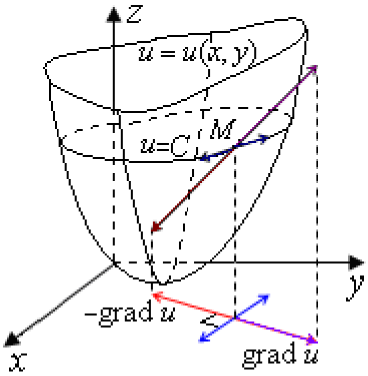


Figure 62.

The directional derivative takes on its greatest negative value if $\varphi = \pi$. Thus, the direction of greatest decrease of $u = u(x, y)$ is the direction opposite to the gradient vector.

Since, for $\varphi = \frac{\pi}{2}$ we obtain $\frac{\partial u}{\partial \vec{a}} = |\text{grad}\ u|\cos\frac{\pi}{2} = 0$, and the derivative along the direction of a vector that is perpendicular to the gradient is zero. That is the scalar field doesn’t change along these directions.

If the gradient vector of scalar field $u = u(x, y)$ is defined in every point of some region D, then we say that ***a vector field of gradients*** is determined in D.

Definition. A vector field that is the gradient of some function is called a ***conservative vector field***. Function u is called a ***scalar potential*** for vector field.

Similarly, the directional derivative and gradient vector are defined in the 3D-space.

Consider a scalar field $u = u(x, y, z)$ defined in a region $D \in \mathbb{R}^3$ and an unit vector $\vec{a} = a_x\vec{i} + a_y\vec{j} + a_z\vec{k}$, then the derivative of u along the direction of the vector $\vec{a}$ is

$$\frac{\partial u}{\partial \vec{a}} = \frac{\partial u}{\partial x}a_x + \frac{\partial u}{\partial y}a_y + \frac{\partial u}{\partial z}a_z, \tag{8.4}$$

and the gradient vector of u

$$\text{grad } u = \frac{\partial u}{\partial x}\vec{i} + \frac{\partial u}{\partial y}\vec{j} + \frac{\partial u}{\partial z}\vec{k}. \tag{8.5}$$

The gradient is in the direction of the **normal** to the level surface passing through the given point.

Note. The symbol ∇ is called ***nabla*** or ***del***. It is used as notation for ***vector differential operator***. In mathematics, an ***operator*** is an action on element of one space to produce element of another space.

Nabla or ***del operator*** acts on the scalar field and gives back a vector

$$\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}.$$

It is used widely in vector calculus and its application in physics and mechanics.

8.4 Divergence and Curl

Let us consider the vector field

$$\vec{F} = P(x, y, z)\vec{i} + Q(x, y, z)\vec{j} + R(x, y, z)\vec{k}$$

defined in a region $D \in \mathbb{R}^3$.

Assume that each component has derivatives with respect to each of the three variables.

Definition. The ***divergence*** of the vector field $\vec{F}$ is the scalar field defined by

$$\text{div}\vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}. \tag{8.6}$$

It could be written in operational form in terms of a dot product

$$\text{div}\vec{F} = \nabla \cdot \vec{F} = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right) \cdot \left(P\vec{i} + Q\vec{j} + R\vec{k}\right) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

What is the physical interpretation of divergence? Let us consider the vector field $\vec{F}$ as the velocity field of a fluid flow. The divergence represents the change in density of the fluid at each point. It shows the tendency of the fluid to diverge from a point.

If the divergence is positive at the point then the fluid flows out of the point and this point is called a ***source*** of the field. Negative divergence means that the fluid moves inward the point. It is called a ***sink*** of the field. Zero divergence means absence of sinks and sources, or they compensate each other.

The absolute value of divergence shows the measure of how much a point is a source or sink. A field which has zero divergence everywhere is called ***solenoidal***.

For example, for the vector field $\vec{F} = x\vec{i} + y\vec{j} + z\vec{k}$ (Fig. 63) the divergence is

$$\mathrm{div}\vec{F} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3\,.$$

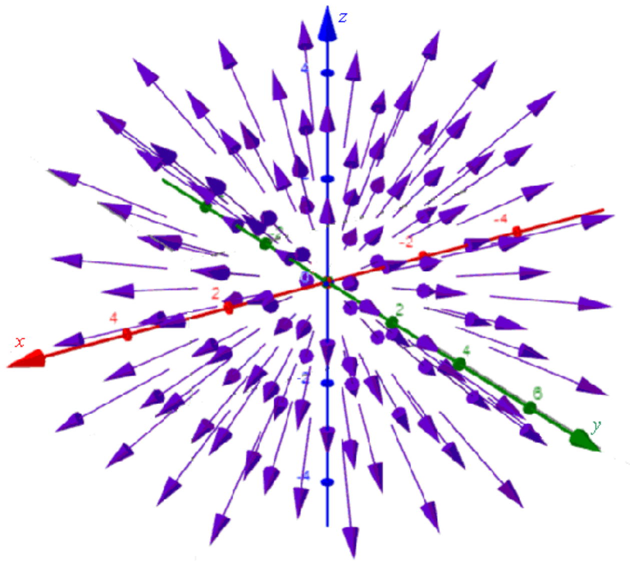


Figure 63.

Since the divergence is a positive constant, each point is a source of the vector field.

For the vector field $\vec{F} = z\vec{i} + x\vec{j} + y\vec{k}$ (Fig. 59) the divergence is

$$\mathrm{div}\vec{F} = \frac{\partial z}{\partial x} + \frac{\partial x}{\partial y} + \frac{\partial y}{\partial z} = 0$$

and this vector field is solenoidal.

Definition. The ***curl*** of the vector field $\vec{F}$ is the vector field defined by

$$\mathrm{rot}\vec{F} = \mathrm{curl}\vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}\,. \qquad (8.7)$$

It could be defined by the cross product

$$\operatorname{rot}\vec{F}=\nabla\times\vec{F}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ P & Q & R\end{vmatrix}=\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)\vec{i}+\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)\vec{j}+\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\vec{k}\,. \quad (8.8)$$

Curl represents the ability of vector field $\vec{F}$ at the given point to rotate about the axis determined by the direction of $\operatorname{rot}\vec{F}$. The magnitude of the curl is the magnitude of rotation. If the curl is zero vector then the vector field $\vec{F}$ is called *irrotational*.

Example. Consider the vector field $\vec{F}=y\vec{i}+(y-x)\vec{j}$ (Fig. 64).

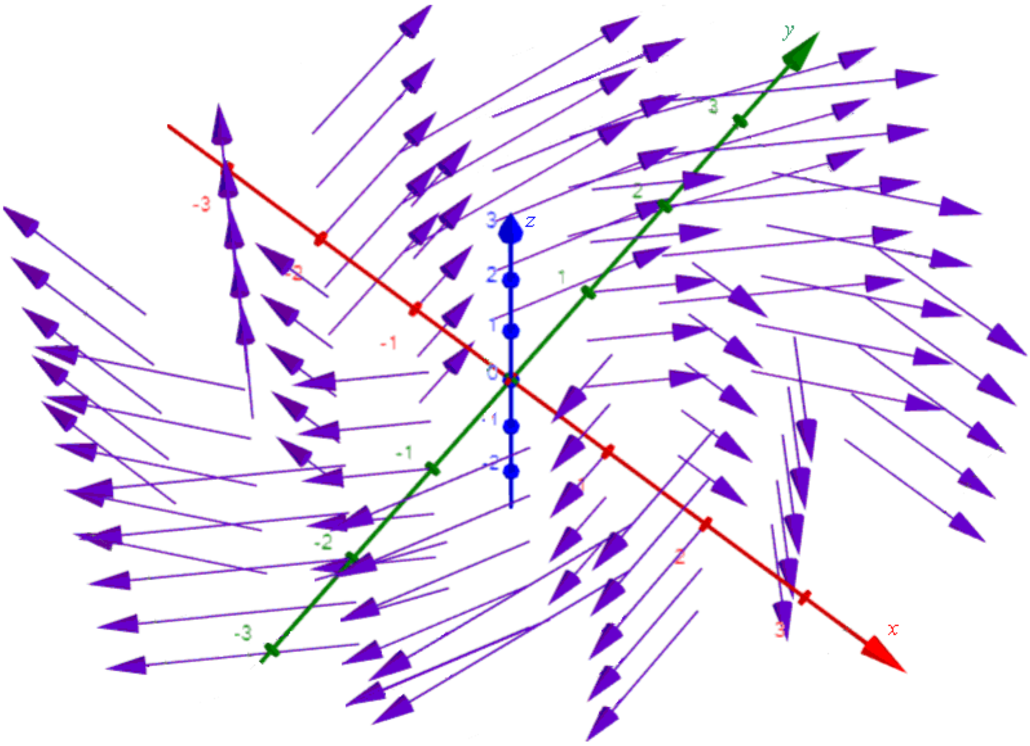


Figure 64.

The curl is

$$\operatorname{rot}\vec{F}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ y & y-x & 0\end{vmatrix}=\left(\frac{\partial 0}{\partial y}-\frac{\partial(y-x)}{\partial z}\right)\vec{i}+\left(\frac{\partial y}{\partial z}-\frac{\partial 0}{\partial x}\right)\vec{j}+\left(\frac{\partial(y-x)}{\partial x}-\frac{\partial y}{\partial y}\right)\vec{k}=-2\vec{k}.$$

The sketch of the curl vector field is on the Figure 65.

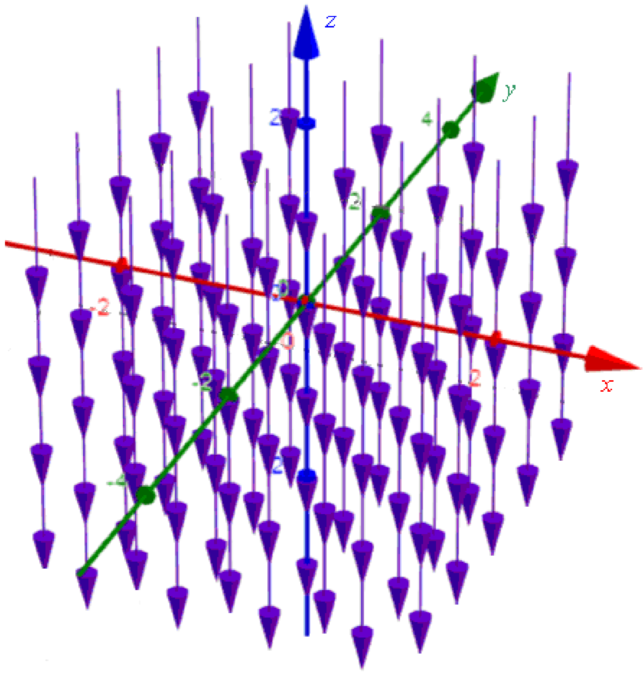


Figure 65.

Consider the scalar field $u = u(x, y, z)$ and corresponding vector field of gradients

$$\operatorname{grad} u = \frac{\partial u}{\partial x}\vec{i} + \frac{\partial u}{\partial y}\vec{j} + \frac{\partial u}{\partial z}\vec{k}.$$

Let us calculate the curl for the vector field of gradients

$$\operatorname{rot}(\operatorname{grad} u) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \end{vmatrix} =$$

$$= \left(\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial y}\right)\right)\vec{i} + \left(\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial z}\right)\right)\vec{j} + \left(\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)\right)\vec{k} = 0$$

Definition. The vector field $\vec{F}$ is called ***conservative*** if $\operatorname{rot}\vec{F} = 0$.

There are a lot of applications of conservative vector fields in mechanics. They represent the forces of physical systems in which energy is conserved, for example gravitational force and the electric force associated to an electrostatic field. Further we are going to learn some properties of conservative vector fields.

9. Line Integrals of Vector Fields

9.1 The Concept of Line Integrals of Vector Fields

I. *The Concept of Line Integrals of Vector Fields*

Consider in 2D space the smooth curve L_{AB} given by

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}, \ a \le t \le b.$$

Assume that the curve is *oriented*, that is a consistent direction is defined along the curve. In addition, the curve is called *simple* if it doesn't cross itself.

Let a point (x, y) move along the curve L_{AB} from the point A to the point B under the action of the force defined by the vector

$$\vec{F} = P(x, y)\vec{i} + Q(x, y)\vec{j}.$$

What is the work W of the force $\vec{F}$ as the point (x, y) moves along the curve L_{AB}?

Divide the curve L_{AB} into n parts by points $A = A_0, \ A_1, \ \ldots, \ A_n = B$ (Fig. 66). Denote the lengths of projection of vectors $\overrightarrow{A_0A_1}$, $\overrightarrow{A_1A_2}, \ldots \overrightarrow{A_{n-1}A_n}$ onto the x axis as follows $\Delta x_1, \ \Delta x_2, \ \ldots, \ \Delta x_n$ and onto the y axis as follows $\Delta y_1, \ \Delta y_2, \ \ldots, \ \Delta y_n$. On each arc $\overset{\cup}{A_{i-1}A_i}$ choose a point $M_i(\xi_i, \eta_i)$ and calculate $P(\xi_i, \eta_i)$ and $Q(\xi_i, \eta_i)$.

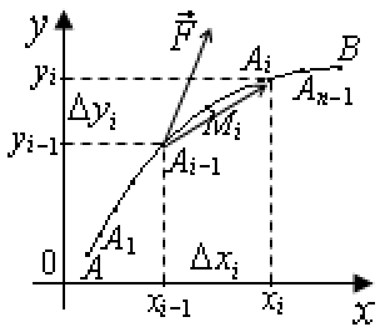


Figure 66.

Since, the work done by a force $\vec{F}$ is the dot product of the force vector $\vec{F_i} = P(\xi_i, \eta_i)\vec{i} + Q(\xi_i, \eta_i)\vec{j}$ and the velocity vector of the point of application $\overrightarrow{\Delta r_i} = \overrightarrow{A_{i-1}A_i} = \Delta x_i \vec{i} + \Delta y_i \vec{j}$, we can write down approximately the work as a sum

$$W \approx \sum_{i=1}^{n} P(\xi_i, \eta_i)\Delta x_i + Q(\xi_i, \eta_i)\Delta y_i = \sum_{i=1}^{n} \left(P(\xi_i, \eta_i)\vec{i} + Q(\xi_i, \eta_i)\vec{j}\right) \cdot \left(\Delta x_i \vec{i} + \Delta y_i \vec{j}\right).$$

If the expression on the right side has a limit as $\Delta x_i \to 0$ and $\Delta y_i \to 0$, then this limit expresses the work of the force $\vec{F}$ over the curve L_{AB}.

Definition. If for continuous on L_{AB} functions $P(x,y)$ and $Q(x,y)$, for any partition of the arc L_{AB} such that $\max\limits_{1\le i\le n}\Delta x_i \to 0$, $\max\limits_{1\le i\le n}\Delta y_i \to 0$ and for any choice of points M_i it exists the limit of integral sum $\sum\limits_{i=1}^{n} P(\xi_i,\eta_i)\Delta x_i + Q(\xi_i,\eta_i)\Delta y_i$, then that limit $\lim\limits_{\substack{\max\limits_{1\le i\le n}\Delta x_i\to 0\\ \max\limits_{1\le i\le n}\Delta y_i\to 0}} \sum\limits_{i=1}^{n} P(\xi_i,\eta_i)\Delta x_i + Q(\xi_i,\eta_i)\Delta y_i$ is called ***the line integral of the vector field over the curve*** L_{AB} and denoted by

$$\int\limits_{L_{AB}} P(x,y)dx + Q(x,y)dy \tag{9.1}$$

or in vector form, in terms of dot product of vectors $\vec{F}$ and $\overrightarrow{dr} = dx\vec{i} + dy\vec{j}$

$$\int\limits_{L_{AB}} \vec{F}\cdot\overrightarrow{dr}\,. \tag{9.2}$$

Similarly, the line integral of the vector field over the curve L_{AB} is determined in 3D-space

$$\int\limits_{L_{AB}} P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz = \int\limits_{L_{AB}} \vec{F}\cdot\overrightarrow{dr}\,. \tag{9.3}$$

where $\overrightarrow{dr} = dx\vec{i} + dy\vec{j} + dz\vec{k}$ and $\vec{F} = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}$.

The curve L_{AB} is called the ***path of integration***.

If curve L_{AB} if closed then we write

$$\oint\limits_{L_{AB}} P(x,y)dx + Q(x,y)dy$$

or

$$\oint\limits_{L_{AB}} P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz\,.$$

In this case we should indicate the orientation of the curve. L_{AB} is called ***positive oriented*** when traveling on it one always has the curve interior to the left (Fig. 67). It is denoted by L_+. In opposite case the curve is called ***negative oriented*** and is denoted by L_-.

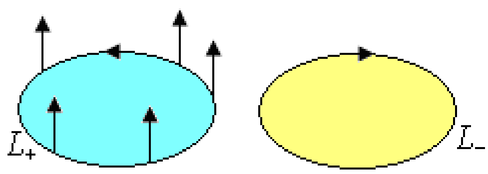


Figure 67.

Properties of a Line Integral

1. Line integral depends on the sense of integration

$$\int_{L_{AB}} P(x,y)dx + Q(x,y)dy = -\int_{L_{BA}} P(x,y)dx + Q(x,y)dy \,.$$

A line integral changes sign when the sense of integration is reversed, since in that case the vector $\overrightarrow{\Delta r_i}$ change the sign: $\overrightarrow{\Delta r_i} = \overrightarrow{A_i A_{i-1}} = -\Delta x_i \vec{i} - \Delta y_i \vec{j}$.

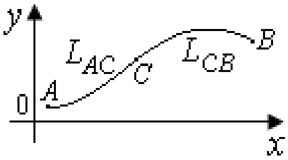


Figure 68.

2. If the curve L_{AB} consists of two parts $L_{AB} = L_{AC} \cup L_{CB}$ (Fig. 68), then

$$\int_{L_{AB}} P(x,y)dx + Q(x,y)dy = \int_{L_{AC}} P(x,y)dx + Q(x,y)dy + \int_{L_{CB}} P(x,y)dx + Q(x,y)dy \,.$$

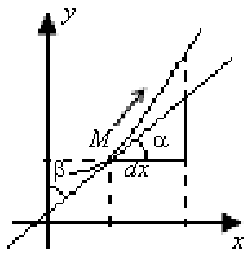


Figure 69.

3. Since $dx = \cos\alpha d\ell$, $dy = \cos\beta d\ell$, where α and β are the angles between the tangent vector and coordinate axes (Fig. 69), we obtain the formula of relationship between linear integral of vector field and linear integral with respect to arc length

$$\int_L Pdx + Qdy = \int_L (P\cos\alpha + Q\cos\beta)d\ell \,. \qquad (9.4)$$

II. ***Evaluating of a Line Integral***

Suppose that $P(x,y)$ and $Q(x,y)$ are continuous real-valued functions defined in some region D that contains the curve L_{AB}. The smooth curve L_{AB} is given by parametric equations $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$, $a \le t \le b$.

Then

$$\overrightarrow{dr} = dx(t)\vec{i} + dy(t)\vec{j} = x'(t)dt\vec{i} + y'(t)dt\vec{j} = (x'(t)\vec{i} + y'(t)\vec{j})dt \,.$$

Hence,

$$\int_{L_{AB}} P(x,y)dx + Q(x,y)dy = \int_{L_{AB}} \vec{F}\cdot\overrightarrow{dr} = \int_a^b \left(P(x,y)\vec{i} + Q(x,y)\vec{j}\right)\cdot\left(x'(t)\vec{i} + y'(t)\vec{j}\right)dt \,.$$

Therefore, general formula for evaluating linear integrals

$$\int_{L_{AB}} P(x,y)dx + Q(x,y)dy = \int_a^b \left(P(x(t),y(t))x'(t) + Q(x(t),y(t))y'(t)\right)dt \,. \qquad (9.5)$$

In 3D space the curve L_{AB} is given by parametric equations $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$, $a \le t \le b$ and

$$\int\limits_{L_{AB}} P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz =$$

$$= \int\limits_a^b \big(P(x(t),y(t),z(t))x'(t) + Q(x(t),y(t),z(t))y'(t) + R(x(t),y(t),z(t))z'(t)\big)dt.$$

Example. Evaluate $\int\limits_L (x+y)dx + (x-y)dy$, if L is a part of circle $(x-1)^2 + (y-1)^2 = 1$ from the point $A(1,0)$ to the point $B(0,1)$ (Fig. 70).

Rewrite the equation of circle in parametrical form

$$\vec{r}(t) = (1+\cos t)\vec{i} + (1+\sin t)\vec{j}, \quad -\frac{\pi}{2} \le t \le \pi.$$

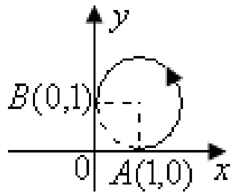


Figure 70.

Then

$$\overrightarrow{dr} = ((1+\cos t)'\vec{i} + (1+\sin t)'\vec{j})dt = (-\sin t\vec{i} + \cos t\vec{j})dt$$

Hence, according to formula (9.5)

$$\int\limits_L (x+y)dx + (x-y)dy = \int\limits_{-\frac{\pi}{2}}^{\pi} (2+\cos t+\sin t)(-\sin t) + (\cos t - \sin t)\cos t)dt =$$

$$= \int\limits_{-\frac{\pi}{2}}^{\pi} (-2\sin t - 2\sin t\cos t + \cos 2t)dt = \left(2\cos t - \sin^2 t + \frac{\sin 2t}{2}\right)\Bigg|_{-\frac{\pi}{2}}^{\pi} = -1.$$

9.2 Green's Formula

Now we are going to understand the relationship between the line integral around a closed curve with a double integral over the region inside the curve.

Green’s Theorem.

Let C be a positively oriented piecewise smooth, simple, closed curve in 2D space and let D be the region enclosed by this curve. If for components of the vector field $\vec{F} = P(x,y)\vec{i} + Q(x,y)\vec{j}$ there exist continuous on D partial derivatives, then

$$\oint\limits_C P(x,y)dx + Q(x,y)dy = \iint\limits_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy\,. \qquad (9.6)$$

Proof.

First, let us evaluate the integral

$$\iint_D \frac{\partial P}{\partial y}dxdy \ .$$

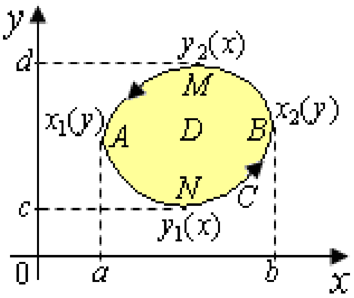


Figure 71.

The region D would be described as follows

$$D = \{(x,y) \mid \ a \le x \le b, \ \ y_1(x) \le y \le y_2(x)\}\ .$$

Then

$$\iint_D \frac{\partial P}{\partial y}dxdy = \int_a^b dx \int_{y_1(x)}^{y_2(x)} \frac{\partial P}{\partial y}dy = \int_a^b P(x,y)\Big|_{y_1(x)}^{y_2(x)}dx = \int_a^b \big(P(x,y_2(x)) - P(x,y_1(x))\big)dx =$$

$$= \int_a^b P(x,y_2(x))dx - \int_a^b P(x,y_1(x))dx$$

Second, we compute

$$\oint_C P(x,y)dx\ .$$

Since, the curve consists of two parts $C = \overset{\cup}{ANB} \cup \overset{\cup}{BMA}$, we have

$$\oint_C P(x,y)dx = \int_{\overset{\cup}{ANB}} P(x,y)dx + \int_{\overset{\cup}{BMA}} P(x,y)dx = \int_a^b P(x,y_1(x))dx + \int_b^a P(x,y_2(x))dx =$$

$$= \int_a^b P(x,y_1(x))dx - \int_a^b P(x,y_2(x))dx = -\left(\int_a^b P(x,y_2(x))dx - \int_a^b P(x,y_1(x))dx\right)$$

Therefore,

$$\oint_C P(x,y)dx = -\iint_D \frac{\partial P}{\partial y}dxdy\ . \tag{9.7}$$

Similarly, we can prove that

$$\oint_C Q(x,y)dy = \iint_D \frac{\partial Q}{\partial x}dxdy\ . \tag{9.8}$$

Indeed,

$$\iint_D \frac{\partial Q}{\partial x}dxdy = \int_c^d dy \int_{x_1(y)}^{x_2(y)} \frac{\partial Q}{\partial x}dx = \int_c^d Q(x,y)\Big|_{x_1(y)}^{x_2(y)}dy = \int_c^d \big(Q(x_2(y),y) - Q(x_1(y),y)\big)dy =$$

$$= \int_c^d Q(x_2(y),y) - \int_c^d Q(x_1(y),y)dy = \int_{\overset{\cup}{NBM}} Q(x,y)dy - \int_{\overset{\cup}{MAN}} Q(x,y)dy = \oint_C Q(x,y)dy\ .$$

The sum of formulas (9.7) and (9.8) gives us the **Green's Formula**

$$\oint_C P(x,y)dx + Q(x,y)dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy \ .$$

Example. Calculate $\oint_L xydx + dy,$ if L is a closed curve, determined by $y = x^2, \ \ y = 1, \ \ x = 0$ (Fig. 72).

There are two ways of solving this problem.

I. According to formula (9.5). Curve L consists of three parts. Therefore,

$$\oint_L xydx + dy = \int_{OA} xydx + dy + \int_{AB} xydx + dy + \int_{BO} xydx + dy \ .$$

For the line OA we have $y = x^2$, $0 \le x \le 1$ and

$$\int_{OA} xydx + dy = \int_{OA} x \cdot x^2 dx + dx^2 = \int_0^1 (x^3 + 2x)dx = \frac{5}{4} .$$

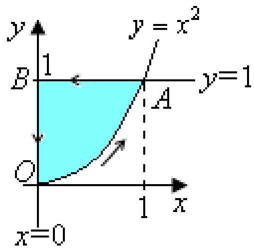


Figure 72.

For the line AB we have $y = 1$, $0 \le x \le 1$ and

$$\int_{AB} xydx + dy = \int_1^0 xdx = -\frac{1}{2} .$$

For the line BO we have $x = 0$, $0 \le y \le 1$ and

$$\int_{BO} xydx + dy = \int_1^0 dy = -1.$$

Finally,

$$\oint_L xydx + dy = \frac{5}{4} - \frac{1}{2} - 1 = -\frac{1}{4} .$$

II. Let us use the Green's formula.

The region D is

$$D = \{(x,y) \,|\ 0 \le x \le 1, \ x^2 \le y \le 1\} .$$

Hence,

$$\oint_L xydx + dy = \iint_D \left(\frac{\partial 1}{\partial x} - \frac{\partial xy}{\partial y} \right) dxdy = -\int_0^1 dx \int_{x^2}^1 xdy = -\int_0^1 xy\Big|_{x^2}^1 dx = -\int_0^1 (x - x^3)dx = -\frac{1}{4} .$$

Both methods give the same answer, but using Green's formula makes the process of solving shorter and easier.

9.3 Independence of Path

I. Consider the line integral

$$\int_{AB} Pdx + Qdy \,,$$

where AB is some plane curve connecting the points A and B, and functions $P(x, y)$ and $Q(x, y)$ are defined in some region D and have continuous partials derivatives in D.

There are infinite number of curves connecting these two points and generally saying the values of integrals depends on the path of integration. But there are some integrals that are dependent only on initial and final positions, not on the path. Such property is called *independence of path*. Let us find out the conditions of independence of path.

Let us consider two arbitrary curves connecting the points A and B: $\overset{\cup}{AMB}$ and $\overset{\cup}{ANB}$ lying in the region D (Fig. 73).

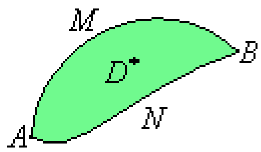


Figure 73.

Let

$$\int_{AMB} Pdx + Qdy = \int_{ANB} Pdx + Qdy \,.$$

Hence,

$$\int_{AMB} Pdx + Qdy - \int_{ANB} Pdx + Qdy = 0 \,.$$

Then, let us change the direction in second integral

$$\int_{AMB} Pdx + Qdy + \int_{BNA} Pdx + Qdy = 0 \,.$$

In this case we obtain the line integral around the closed curve

$$\int_{AMBNA} Pdx + Qdy = \int_{L} Pdx + Qdy = 0 \,. \tag{9.9}$$

The question arises: what conditions must the functions $P(x, y)$ and $Q(x, y)$ satisfy in order that the line integral $\oint_L Pdx + Qdy$ along any closed contour L be equal to zero.

Let the closed curve L be a boundary of the region D^* and let us use the Green's formula to the line integral from formula (9.9).

Hence,

$$\oint_L Pdx + Qdy = \iint_{D^*} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy = 0 .$$

Naturally, we have the conditions

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0 \quad \Rightarrow \quad \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} .$$

As a result, we have the equivalent statements.

1. The line integral $\int_{AB} Pdx + Qdy$ ***is independent of path if the line integral*** $\oint_L Pdx + Qdy$ ***along any closed contour*** L ***be equal to zero.***

2. The line integral $\int_{AB} Pdx + Qdy$ ***is independent of path if*** $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ ***at all points of domain*** D***, such that points*** A ***and*** B ***lying in*** D***.***

II. Consider the conservative vector field $\vec{F} = P(x, y)\vec{i} + Q(x, y)\vec{j}$. According to definition there exists the scalar field (scalar potential) $u = u(x, y)$ that is

$$\vec{F} = \operatorname{grad} u = \frac{\partial u}{\partial x}\vec{i} + \frac{\partial u}{\partial y}\vec{j} .$$

Therefore,

$$\frac{\partial u}{\partial x} = P(x, y), \qquad \frac{\partial u}{\partial y} = Q(x, y) .$$

Let us find second partial derivatives of $u = u(x, y)$ with respect to x and y

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial P}{\partial y} \quad \text{and} \quad \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial Q}{\partial x} .$$

Since, $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$, we have $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.

Definition. Expression $P(x, y)dx + Q(x, y)dy$ is called ***an exact differential of some function*** $u = u(x, y)$ if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$. It is denoted by

$$du(x, y) = P(x, y)dx + Q(x, y)dy . \tag{9.10}$$

In this case the line integral $\int\limits_{AB} Pdx + Qdy$ along any curve connecting the points A and B is equal to the differences between the values of the function $u = u(x, y)$ at these points

$$\int\limits_A^B Pdx + Qdy = \int\limits_A^B du(x, y) = u(B) - u(A)\,. \tag{9.11}$$

III. According to definition and conditions of independence of path we can make a conclusion that the line integral of the vector field is independent of path only for conservative vector fields.

The same idea we apply for line integrals of the vector field in 3D space.

Consider the line integral

$$\int\limits_{AB} P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz\,,$$

where $\vec{F} = P(x, y, z)\vec{i} + Q(x, y, z)\vec{j} + R(x, y, z)\vec{k}$ is conservative, that is $\mathrm{rot}\vec{F} = 0$.

Since, $\mathrm{rot}\vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k} = 0$, we have the conditions of independence of path in 3D space

$$\frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \quad \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}, \tag{9.12}$$

and

$$\int\limits_A^B Pdx + Qdy + Rdz = \int\limits_A^B du(x, y, z) = u(B) - u(A)\,. \tag{9.13}$$

Formulas (9.11) and (9.13) are called ***fundamental theorem for line integrals*** and could be written in vector form if the curve is defined $\vec{r} = \vec{r}(t)$, $a \le t \le b$

$$\int\limits_L \nabla u \cdot d\vec{r} = u(\vec{r}(b)) - u(\vec{r}(a))\,. \tag{9.14}$$

The above result is valid for the functions of any number of variables.

9.4 Application of a Line Integrals of Vector Fields

I. ***The Work of the Force Along the Curve***

Let a point move along the curve L_{AB} from the point A to the point B under the action of the force $\vec{F}$, then the work W of the force $\vec{F}$ along the curve L_{AB} is

$$W = \int_{L_{AB}} \vec{F} \cdot d\vec{r} \,. \tag{9.15}$$

If the force is conservative force then the total work done along the curve is independent of the path and if the curve is closed then the work is zero.

Example. Find the work of the force $\vec{F} = yz\vec{i} + xz\vec{j} + xy\vec{k}$ along the straight line from the point $(1,1,1)$ to the point $(2,3,4)$.

First, we find the equation of the straight line

$$\frac{x-1}{2-1} = \frac{y-1}{3-1} = \frac{z-1}{4-1} \quad \Rightarrow \quad \frac{x-1}{1} = \frac{y-1}{2} = \frac{z-1}{3}$$

in parametrical form

$$\begin{cases} x = 1 + t, \\ y = 1 + 2t, \qquad 0 \le t \le 1. \\ z = 1 + 3t, \end{cases}$$

Then the work is

$$W = \int_L yzdx + xzdy + xydz = \int_0^1 [(1+2t)(1+3t) + (1+t)(1+3t)\cdot 2 + (1+t)(1+2t)\cdot 3]dt =$$

$$= \int_0^1 (18t^2 + 22t + 6)dt = 23 \text{ (units of work).}$$

II. ***Finding Potentials for Conservative Fields***

1. Consider the conservative vector field

$$\vec{F} = P(x,y)\vec{i} + Q(x,y)\vec{j} \,.$$

There exists the scalar field (***potential***) $u = u(x,y)$ such that the exact differential of the function $u = u(x,y)$ is

$$du(x,y) = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = P(x,y)dx + Q(x,y)dy \,.$$

In this case the potential could be found by formula

$$u(x,y)=\int_{(x_0,y_0)}^{(x,y)} P(x,y)dx+Q(x,y)dy=\int_{x_0}^{x} P(x,y_0)dx+\int_{y_0}^{y} Q(x,y)dy\,. \tag{9.16}$$

2. For the three-dimensional conservative vector field

$$\vec{F}=P(x,y,z)\vec{i}+Q(x,y,z)\vec{j}+R(x,y,z)\vec{k}$$

there exists the scalar field (***potential***) $u=u(x,y,z)$

$$u(x,y,z)=\int_{(x_0,y_0,z_0)}^{(x,y,z)} P(x,y,z)dx+Q(x,y,z)dy+R(x,y,z)dz=$$

$$=\int_{x_0}^{x} P(x,y_0,z_0)dx+\int_{y_0}^{y} Q(x,y,z_0)dy+\int_{z_0}^{z} R(x,y,z)dz\,. \tag{9.17}$$

Note. In many cases we can choose $(x_0,y_0,z_0)=(0,0,0)$.

Example. State that the vector field

$$\vec{F}=(3x^2+2xy)\vec{i}+(x^2+2y+z)\vec{j}+(y+3z^2)\vec{k}$$

is conservative and find the potential $u(x,y,z)$.

Since, $\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}=2x,\quad \frac{\partial R}{\partial y}=\frac{\partial Q}{\partial z}=1,\quad \frac{\partial P}{\partial z}=\frac{\partial R}{\partial x}=0,$ the vector field $\vec{F}$ is conservative and

$$du=(3x^2+2xy)dx+(x^2+2y+z)dy+(y+3z^2)dz\,.$$

Hence, using (9.17) when $(x_0,y_0,z_0)=(0,0,0)$:

$$u(x,y,z)=\int_{(0,0,0)}^{(x,y,z)} (3x^2+2xy)dx+(x^2+2y+z)dy+(y+3z^2)dz+C=$$

$$=\int_0^x 3x^2dx+\int_0^y (x^2+2y)dy+\int_0^z (y+3z^2)dz=x^3+x^2y+y^2+yz+z^3+C.$$

III. *Area of a 2D Region*

Consider the region D and the closed curve C is a boundary of D. Then the area of the region D could be calculated by liner integral as follows

$$S=\frac{1}{2}\oint_C -ydx+xdy\,. \tag{9.18}$$

10. Surface Integral of Vector Fields

10.1 The Concept of a Surface Integral of Vector Fields

I. *The Concept of Oriented Surface*

Consider a smooth surface S and a closed contour L lying on this surface (Fig. 74). Plot the unit orthogonal vector $\vec{N}_P$ at the point P of the contour L. Obviously, it is normal vector to the surface S. Let us move the normal vector $\vec{N}_P$ along the curve L.

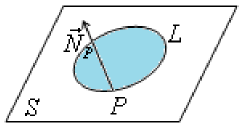
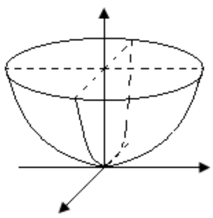


Figure 74.

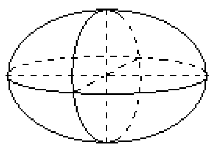
When returning at the point P, it could be obtained the same vector $\vec{N}_P$ or the opposite vector $-\vec{N}_P$. If the vector doesn't change the direction then the surface is called ***two-sided***. Otherwise the surface is called ***one-sided***.

The two-sided surfaces are called ***oriented***. That means that surface has two sets of normal vectors. The set that we choose give us the surface an orientation. This choice is important when we have to compute surface integrals of vector fields.

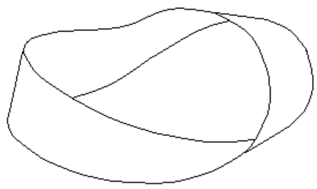
Most of the surfaces are oriented. For example, paraboloid, ellipsoid (Fig. 75). Mobius strip is an example of one-sided surface.



Paraboloid



Ellipsoid



Mobius Strip

Figure 75.

The closed surface is called ***positively oriented*** if we choose the set of unit normal vectors that point **outward (outer normal)**. Otherwise we say that the surface has ***negative orientation*** (unit normal vectors point **inward (inner normal)**) (Fig 76).

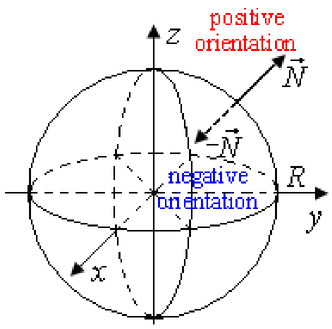


Figure 76.

Since the orientation of surface is defined by normal vector, we should know the ways of determining normal.

Let the surface S is determined as follows $F(x,y,z)=0$. Then the gradient vector

$$\operatorname{grad} F = \left(\frac{\partial F(x,y,z)}{\partial x}, \frac{\partial F(x,y,z)}{\partial y}, \frac{\partial F(x,y,z)}{\partial z} \right) = \left(F'_x, F'_y, F'_z \right)$$

is orthogonal to the surface S.

Finding the corresponding unit vector

$$\vec{n} = \frac{\operatorname{grad} F}{|\operatorname{grad} F|} = \left(\frac{F'_x}{\sqrt{F'^2_x + F'^2_y + F'^2_z}}, \frac{F'_y}{\sqrt{F'^2_x + F'^2_y + F'^2_z}}, \frac{F'_z}{\sqrt{F'^2_x + F'^2_y + F'^2_z}} \right) \tag{10.1}$$

we determine the normal vector.

Here

$$\cos\alpha = \frac{F'_x}{\sqrt{F'^2_x + F'^2_y + F'^2_z}},$$

$$\cos\beta = \frac{F'_y}{\sqrt{F'^2_x + F'^2_y + F'^2_z}},$$

$$\cos\gamma = \frac{F'_z}{\sqrt{F'^2_x + F'^2_y + F'^2_z}}$$

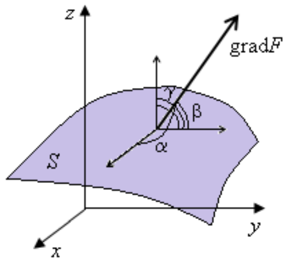


Figure 77.

are the cosines of the angles between the vector and coordinate axes (***directional cosines***).

If the surface is defined by the equation $z = z(x,y)$, then $F(x,y,z) = z(x,y) - z = 0$ and directional cosines are

$$\cos\alpha = \frac{z'_x}{\sqrt{z'^2_x + z'^2_y + 1}}, \quad \cos\beta = \frac{z'_y}{\sqrt{z'^2_x + z'^2_y + 1}}, \quad \cos\gamma = \frac{-1}{\sqrt{z'^2_x + z'^2_y + 1}}. \tag{10.2}$$

If the surface is described in parametric form

$$\vec{r}(u,v) = \left(x(u,v), y(u,v), z(u,v) \right), \quad (u,v) \in D,$$

then

$$\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} = \frac{J_1(u,v)\vec{i} + J_2(u,v)\vec{j} + J_3(u,v)\vec{k}}{\sqrt{J_1^2(u,v) + J_2^2(u,v) + J_3^2(u,v)}}, \tag{10.3}$$

where

$$J_1(u,v) = \begin{vmatrix} y'_u & z'_u \\ y'_v & z'_v \end{vmatrix}, \quad J_2(u,v) = \begin{vmatrix} z'_u & x'_u \\ z'_v & x'_v \end{vmatrix}, \quad J_3(u,v) = \begin{vmatrix} x'_u & y'_u \\ x'_v & y'_v \end{vmatrix}.$$

II. ***The Surface Integral of Vector Field***

Let an oriented surface S be given in 3D space and suppose that unit vector normal to that surface is $\vec{n} = \cos\alpha\vec{i} + \cos\beta\vec{j} + \cos\gamma\vec{k}$.

Consider the vector field

$$\vec{F} = P(x, y, z)\vec{i} + Q(x, y, z)\vec{j} + R(x, y, z)\vec{k},$$

where $P(x, y, z)$, $Q(x, y, z)$ and $R(x, y, z)$ are continuous functions over the surface S.

If the components of $\vec{n}$ are continuous functions of x, y and z, then a dot product $\vec{F}\cdot\vec{n}$ is a continuous scalar function we may define the surface integral over the surface S

$$\iint_S \vec{F}\cdot\vec{n}\,d\sigma = \iint_S \big(P(x, y, z)\cos\alpha + Q(x, y, z)\cos\beta + R(x, y, z)\cos\gamma\big)d\sigma. \qquad (10.4)$$

Since $dxdy = \cos\gamma d\sigma$, $dxdz = \cos\beta d\sigma$ and $dydz = \cos\alpha d\sigma$, we rewrite the integral as follows

$$\iint_S \vec{F}\cdot\vec{n}\,d\sigma = \iint_S P(x, y, z)dydz + Q(x, y, z)dxdz + R(x, y, z)dxdy. \qquad (10.5)$$

Definition. Integral defined by formula (10.4) is called ***the surface integral of vector field*** or ***flux integral of*** $\vec{F}$ ***over*** S.

III. ***Evaluating the Surface Integral of Vector Field***

Formula (10.4) expressed the relationship between the surface integral over the surface and the flux integral. Therefore, we may evaluate the surface integral of vector field in terms of the surface integral over the surface. Methods was shown in chapter 7.2 of this issue.

Example. Evaluate $\iint_S (x - y + 1.5z)dydz + xdzdx - zdxdy$, where S is an external side of the portion of the plane $2x - 2y + z - 2 = 0$ bounded by the coordinate planes.

Let us calculate the normal vector $\vec{N} = 2\vec{i} - 2\vec{j} + \vec{k}$. Corresponding unit vector is

$$\vec{n} = \frac{2\vec{i} - 2\vec{j} + \vec{k}}{\sqrt{2^2 + (-2)^2 + 1^2}} = \frac{2\vec{i} - 2\vec{j} + \vec{k}}{3}.$$

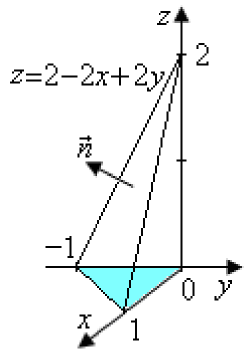


Figure 78.

Hence,

$$I=\iint_S (x-y+1.5z)dydz+xdzdx-zdxdy=\iint_S\left((x-y+1.5z)\frac{2}{3}+x\left(-\frac{2}{3}\right)-z\frac{1}{3}\right)d\sigma .$$

Since $d\sigma=\sqrt{2^2+(-2)^2+1^2}dxdy=3dxdy$ and the projection of S on xy-plane is the region $D=\{(x,y)\,|\,0\le x\le 1, x-1\le y\le 0\}$, we have

$$I=\iint_D ((x-y+1{,}5(-2x+2y+2))2+x(-2)-(-2x+2y+2))dxdy=\iint_D (-4x+2y+4)dxdy=$$

$$=\int_0^1 dx\int_{x-1}^0(-4x+2y+4)dy=\int_0^1\left(-4xy+y^2+4y\right)\Big|_{x-1}^0 dx=1.$$

Note. For the closed surfaces we use Divergence Theorem.

10.2 Divergence Theorem

Consider a solid V in 3D space and S is the boundary surface of V with positive orientation. Let $\vec{F}=P(x,y,z)\vec{i}+Q(x,y,z)\vec{j}+R(x,y,z)\vec{k}$ be a vector field whose components have continuous first order partial derivatives. Then,

$$\oiint_S Pdydz+Qdzdx+Rdxdy=\iiint_V\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)dxdydz$$

or in vector form

$$\oiint_S \vec{F}\cdot\vec{n}\,d\sigma=\iiint_V \operatorname{div}\vec{F}dxdydz .$$

Proof.

Let a solid $V\subset R^3$ is projected on xy-plane into a regular 2D domain D. Assume that S consists of three parts (Fig. 79):

S_1 is a lower part and its equation is $z=z_1(x,y)$,

S_2 is a upper part and its equation is $z=z_2(x,y)$,

S_3 is a cylindrical surface with generator parallel to the z-axis.

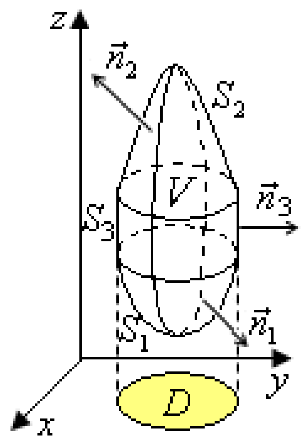


Figure 79.

Consider the integral

$$\iiint\limits_V \frac{\partial R(x,y,z)}{\partial z}dxdydz \ .$$

Let us evaluate that integral

$$\iiint\limits_V \frac{\partial R(x,y,z)}{\partial z}dxdydz = \iint\limits_D dxdy \int\limits_{z_1(x,y)}^{z_2(x,y)} \frac{\partial R(x,y,z)}{\partial z}dz =$$

$$= \iint\limits_D R(x,y,z_2(x,y))dxdy - \iint\limits_D R(x,y,z_1(x,y))dxdy \ . \tag{10.6}$$

Since, $\cos(\vec{n}_2, z)$ is positive on S_2, $\cos(\vec{n}_1, z)$ is negative on S_1 and $\cos(\vec{n}_3, z) = 0$ on the surface S_3, the double integrals in the right side of (10.6) are equal to the corresponding surface integrals:

$$\iint\limits_D R(x,y,z_2(x,y))dxdy = \iint\limits_{S_2} R(x,y,z)\cos(\vec{n}_2, z)d\sigma = \iint\limits_{S_2} R(x,y,z)dxdy \ , \tag{10.7}$$

$$\iint\limits_D R(x,y,z_1(x,y))dxdy = \iint\limits_{S_1} R(x,y,z)\cos(\vec{n}_1, z)d\sigma = \iint\limits_{S_1} R(x,y,z)dxdy \ , \tag{10.8}$$

$$\iint\limits_{S_3} R(x,y,z)dxdy = \iint\limits_{S_3} R(x,y,z)\cos(\vec{n}_3, z)d\sigma = 0 \ .$$

Hence, putting (10.7) and (10.8) into (10.6), we have

$$\iint\limits_D R(x,y,z_2(x,y))dxdy - \iint\limits_D R(x,y,z_1(x,y))dxdy =$$

$$= \iint\limits_{S_2} R(x,y,z)dxdy - \iint\limits_{S_1} R(x,y,z)dxdy + \iint\limits_{S_3} R(x,y,z)dxdy =$$

$$= \iint\limits_{S_2 \cup S_1 \cup S_3} R(x,y,z)dxdy = \oint\!\!\!\oint\limits_S R(x,y,z)dxdy \ .$$

Therefore,

$$\iiint\limits_V \frac{\partial R(x,y,z)}{\partial z}dxdydz = \oint\!\!\!\oint\limits_S R(x,y,z)dxdy \ . \tag{10.9}$$

Analogously, we obtain the equalities

$$\iiint\limits_V \frac{\partial P(x,y,z)}{\partial x}dxdydz = \oint\!\!\!\oint\limits_S P(x,y,z)dydz \ , \tag{10.10}$$

$$\iiint\limits_V \frac{\partial Q(x,y,z)}{\partial y}dxdydz = \oint\!\!\!\oint\limits_S Q(x,y,z)dzdx \ . \tag{10.11}$$

Adding together equalities (10.9)-(10.11) term by term, we get the statement of Divergence Theorem.

Example. Evaluate the integral

$$\oiint\limits_S xzdydz + xydzdx + yzdxdy \ ,$$

where S is an outer side of the closed surface lying in the first octant and consists of cylinder $x^2 + y^2 = R^2$ and planes $x = 0,\ y = 0,\ z = 0,\ z = H$ (Fig. 80).

In this case $P = xz,\ Q = xy,\ R = yz.$ Since the surface is closed, let us use the Divergence Theorem

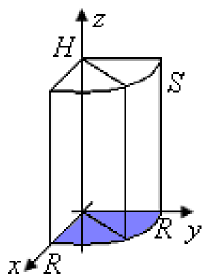


Figure 80.

$$\oiint\limits_S xzdydz + xydzdx + yzdxdy = \iiint\limits_V \left(\frac{\partial xz}{dx} + \frac{\partial xy}{dy} + \frac{\partial yz}{dz} \right) dxdydz =$$

$$= \iiint\limits_V (z + x + y)dxdydz = \iint\limits_{D_{xy}} dxdy \int\limits_0^H (x + y + z)dz =$$

$$= H \iint\limits_{D_{xy}} \left(x + y + \frac{H}{2} \right) dxdy = \begin{vmatrix} x = \rho\cos\varphi, & y = \rho\sin\varphi, \\ 0 \le \varphi \le \dfrac{\pi}{2}, & 0 \le \rho \le R \end{vmatrix} =$$

$$= H \int\limits_0^{\frac{\pi}{2}} d\varphi \int\limits_0^R \left(\rho\cos\varphi + \rho\sin\varphi + \frac{H}{2} \right) \rho d\rho = HR^2 \left(\frac{2R}{3} + \frac{\pi H}{8} \right).$$

10.3 Stokes' Formula

Let us state the relationship between surface integral of the vector field and the linear integral of the vector field.

Consider the smooth bounded surface S such that any straight line parallel to the axis cuts it in one point. The surface is defined by the equation $z = z(x, y)$. It has continuous first partial derivatives. The direction of the normal vector $\vec{n}$ is positive ($\cos(\vec{n}, z) > 0$). Assume that the projection of S onto xy-plane is region D (Fig. 81).

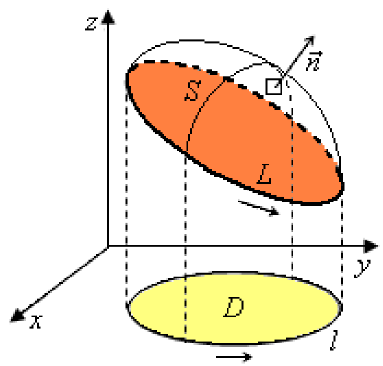


Figure 81.

Let L be the boundary of the surface S, l be its projection of S onto xy-plane, that means it is a curve that bounds a region D. We consider the positive direction along l as that which corresponds to the positive direction along L.

The directional cosines of the normal are expressed as follows

$$\cos\alpha = \frac{-z'_x}{\sqrt{z'^2_x + z'^2_y + 1}}, \quad \cos\beta = \frac{-z'_y}{\sqrt{z'^2_x + z'^2_y + 1}}, \quad \cos\gamma = \frac{1}{\sqrt{z'^2_x + z'^2_y + 1}}. \qquad (10.12)$$

Let $\vec{F} = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}$ be a vector field whose components have continuous first order partial derivatives.

Evaluate the line integral of vector field along the closed contour L

$$\oint_L P(x,y,z)dx.$$

Since L belongs to the surface S, it could be defined by the equation $z = z(x,y)$, where (x,y) are the points of the curve l.

Thus,

$$\frac{\partial}{\partial y} P(x,y,z(x,y)) = P_y + P_z \cdot z_y.$$

Let us use Green's formula

$$\oint_L P(x,y,z)dx = \oint_L P(x,y,z(x,y))dx = -\iint_D \left(P_y + P_z \cdot z_y\right)dxdy.$$

From formulas (10.12) we have $\dfrac{\cos\beta}{\cos\gamma} = -z_y$ and, therefore,

$$-\iint_D \left(P_y + P_z \cdot z_y\right)dxdy = -\iint_D \left(P_y - P_z \cdot \frac{\cos\beta}{\cos\gamma}\right)dxdy =$$

$$= -\iint_S \left(P_y - P_z \cdot \frac{\cos\beta}{\cos\gamma}\right)\cos\gamma\, d\sigma = \iint_S \left(\frac{\partial P}{\partial z}\cos\beta - \frac{\partial P}{\partial y}\cos\gamma\right)d\sigma.$$

Hence

$$\oint_L P(x,y,z)dx = \iint_S \left(\frac{\partial P}{\partial z}\cos\beta - \frac{\partial P}{\partial y}\cos\gamma\right)d\sigma. \qquad (10.13)$$

Similarly

$$\oint_L Q(x,y,z)dy = \iint_S \left(\frac{\partial Q}{\partial x}\cos\gamma - \frac{\partial Q}{\partial z}\cos\alpha\right)d\sigma, \qquad (10.14)$$

$$\oint_L R(x,y,z)dz = \iint_S \left(\frac{\partial R}{\partial y}\cos\alpha - \frac{\partial R}{\partial x}\cos\beta \right) d\sigma . \qquad (10.15)$$

Adding the left and right sides of the equalities (10.13)-(10.15), we get

$$\oint_L P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz =$$

$$= \iint_S \left[\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right)\cos\alpha + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right)\cos\beta + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)\cos\gamma \right] d\sigma =$$

$$= \iint_S \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) dydz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) dzdx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy .$$

This formula is called **Stokes' Formula**.

It could be written in vector form as

$$\oint_L \vec{F} \cdot d\vec{r} = \iint_S \operatorname{rot}\vec{F} \cdot \vec{n} d\sigma .$$

Example. Evaluate the line integral $\oint_{L_+} (z^2 - x^2)dx + (x^2 - y^2)dy + (y^2 - z^2)dy$, where L is circle $x^2 + y^2 = 4$, $z = 2$.

Let us use Stokes' Formula.

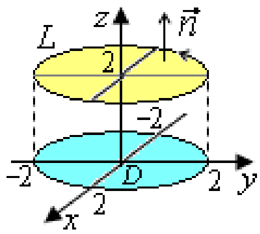


Figure 82.

Consider as the surface S the portion of the plane $z = 2$ bounded by circle L. The projection of S onto xy-plane is a circle $x^2 + y^2 = 4$. The directional cosines of the normal vector are

$$\cos\alpha = 0, \ \cos\beta = 0, \ \cos\gamma = 1.$$

Let us find the curl of the vector field $\vec{F} = (z^2 - x^2)\vec{i} + (x^2 - y^2)\vec{j} + (y^2 - z^2)\vec{k}$:

$$\operatorname{rot}\vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)\vec{k} = 2y\vec{i} + 2z\vec{j} + 2x\vec{k} .$$

According to Stokes' Formula, we obtain

$$\oint_{L_+} (z^2 - x^2)dx + (x^2 - y^2)dy + (y^2 - z^2)dy = \iint_S \operatorname{rot}\vec{F} \cdot \vec{n} d\sigma = \iint_S (2y\vec{i} + 2z\vec{j} + 2x\vec{k}) \cdot \vec{k} d\sigma =$$

$$= \iint_D 2x dxdy = \left| \begin{matrix} x = \rho\cos\varphi, \ \ y = \rho\sin\varphi \\ 0 \le \varphi \le 2\pi, \ \ 0 \le \rho \le 2 \end{matrix} \right| = 2\int_0^{2\pi} \cos\varphi d\varphi \int_0^2 \rho^2 d\rho = 0 .$$

10.4 Flux and Circulation of a Vector Field

Let us look at two important physical application of surface and line integrals of the vector field: flux and circulation.

Consider the vector field

$$\vec{F} = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k},$$

where P, Q, R are continuous differentiable functions in some region Ω in 3D space. Physically vector $\vec{F}$ is the velocity vector of a liquid flowing in the region Ω.

I. *Flux of a Vector Field*

Let S be the surface placed somewhere in Ω. Let us find the flux of $\vec{F}$ through S.

Definition. The flux of the vector field through the surface is a total amount of liquid passing through the surface per unit time in the direction of normal vector.

Look at some cases.

Case 1. Let S is a portion of plane that is perpendicular to the constant vector field $\vec{F}$. That is the normal $\vec{n}$ is parallel to $\vec{F}$ at any point of S (Fig. 83).

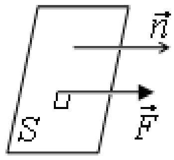


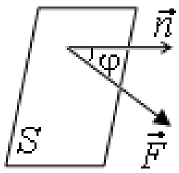
Figure 83.

Any part of the fluid has the same velocity v. In time t the volume of fluid is the volume of the box of width vt and area of the base A_S.

According to definition the flux of $\vec{F}$ through the surface is velocity times the area

$$\text{Flux}_S \vec{F} = \left|\vec{F}\right| \cdot A_S.$$

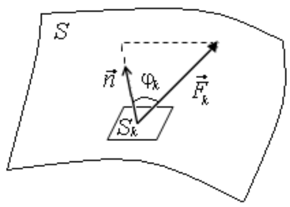
Case 2. Let now the portion of plane is tilted with respect to the direction of flow (Fig. 84). The flux depends on the angle φ between the vector field and the normal



$$\text{Flux}_S \vec{F} = \left|\vec{F}\right| A_S \cos\varphi = \vec{F} \cdot \vec{n}.$$

Figure 84.

Case 3. Consider some arbitrary oriented surface S. Partition the surface into n subregions S_k, $1 \le k \le n$ (Fig. 85). The flux through each part is defined as in the case 2:



$$\left|\vec{F}_k\right| A_{S_k} \cos\varphi_k = \vec{F}_k \cdot \vec{n}_k.$$

Figure 85.

Thus,

$$\text{Flux}_S \vec{F} \approx \sum_{k=1}^{n} \vec{F}_k \cdot \vec{n}_k \,. \tag{10.16}$$

As n tends to infinity then the sum (10.16) leads to the surface integral and give us **the flux of the vector field $\vec{F}$ through the surface S**

$$\text{Flux}_S \vec{F} = \iint_S P(x,y,z)dydz + Q(x,y,z)dzdx + R(x,y,z)dxdy = \iint_S \vec{F} \cdot \vec{n} d\sigma \,. \tag{10.17}$$

If the surface S is closed then the flux could be written in terms of Divergence Theorem

$$\text{Flux}_S \vec{F} = \oiint_S \vec{F} \cdot \vec{n} d\sigma = \iiint_V \text{div}\vec{F} dv \,. \tag{10.18}$$

It yields the quantity of liquid flowing out (or into) of the domain through the surface in unit time. If $\text{div}\vec{F} = 0$, then the quantity of liquid flowing out (or into) is zero. That is the quantity of liquid flowing out is equal to the quantity of liquid flowing in or there are no sources of liquid outside (or inside) the surface.

As examples, let us consider electric and magnetic fields.

An electric field $\vec{E}$ surrounds an electric charge. For example, positive point charge is represented by radially outward vectors (Fig. 86). It is conservative vector field.

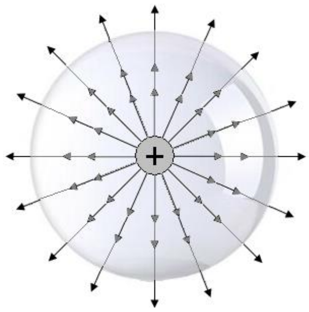


Figure 86.

The electric flux through any closed surface S is proportional to the net electric charge inside

$$\oiint_S \vec{E} \cdot \vec{n} d\sigma = \frac{q}{\varepsilon_0} \,,$$

where q is the total electric charge inside the surface S and ε_0 is the electric constant.

For instance, magnetic field $\vec{B}$ is a solenoidal vector field. It is well-known from physics that the magnetic flux through any closed surface (Fig. 87) is always equal to zero

$$\oiint_S \vec{B} \cdot \vec{n} d\sigma = 0 \,.$$

That property is true for any solenoidal field $\vec{F}$, since $\text{div}\vec{F} = 0$.

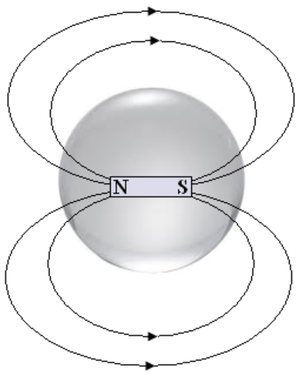


Figure 87.

II. ***Circulation of a Vector Field***

Let L is an oriented closed curve in Ω.

Definition. The circulation of the vector field around the contour is the line integral of the vector field taken along the closed curve

$$\text{Circulation}_L \vec{F} = \oint_L P dx + Q dy + R dz = \oint_L \vec{F} \cdot d\vec{r} .$$

Due to Stokes' formula, we could rewrite it as follows

$$\text{Circulation}_L \vec{F} = \oint_L \vec{F} \cdot d\vec{r} = \iint_S \text{rot}\vec{F} \cdot \vec{n} d\sigma .$$

Thus, the circulation of a vector field around the contour of some surface is equal to the flux of the curl through this surface.

Circulation measures how much the vector field is aligned with the contour. Physically the line integral indicates how much the vector field tends to circulate around the contour.

Let us return to the examples of electric and magnetic fields.

Since electric field $\vec{E}$ is conservative, the curl of that field is equal zero: $\text{rot}\vec{E} = 0$.

Hence,

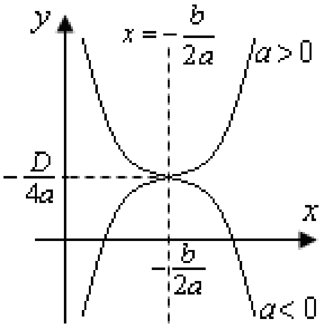
$$\text{Circulation}_L \vec{E} = \iint_S \text{rot}\vec{E} \cdot \vec{n} d\sigma = 0 .$$

For any conservative vector field $\vec{F}$ we have $\text{Circulation}_L \vec{F} = 0$ and it have no ability to circulate (irrotational vector field).

Otherwise, for the magnetic field the line integral around the contour is proportional to the electric current I passing through the path

$$\text{Circulation}_L \vec{B} = \oint_L \vec{B} \cdot d\vec{r} = \mu_0 I .$$

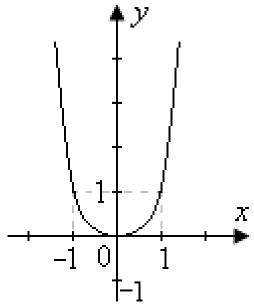
Appendix 1. Graphs of Certain Functions



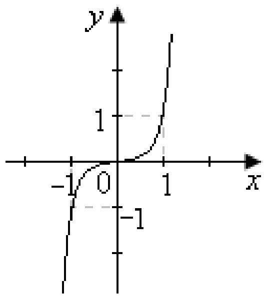
$y = ax^2 + bx + c,\ a \neq 0$

(Parabola)

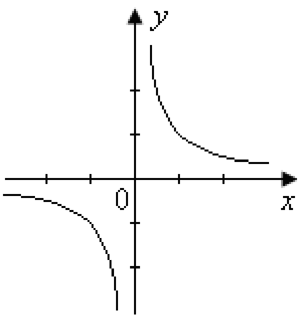
Power Function



$y = x^n$, ***n*** is even



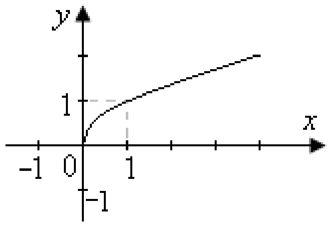
$y = x^n$, ***n*** is odd



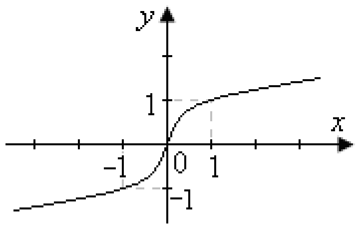
Hyperbola

$y = \frac{k}{x}$

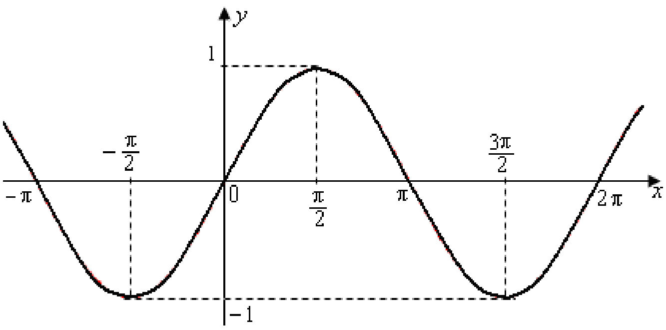
Inverse Power Functions



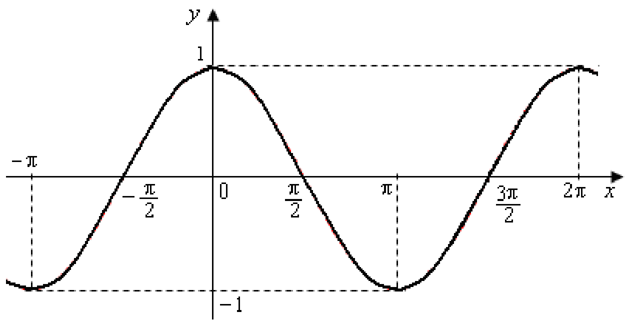
$y = \sqrt[n]{x}$, ***n*** is even



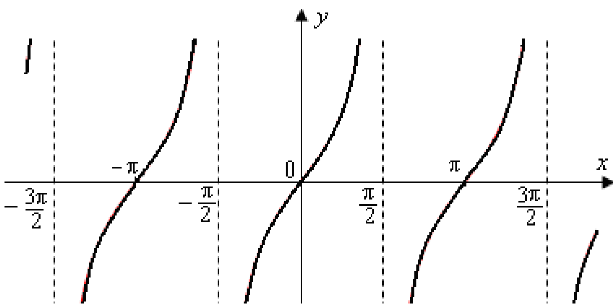
$y = \sqrt[n]{x}$, ***n*** is odd



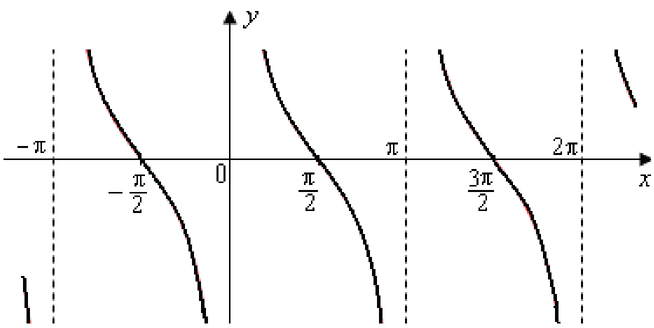
$y = \sin x$



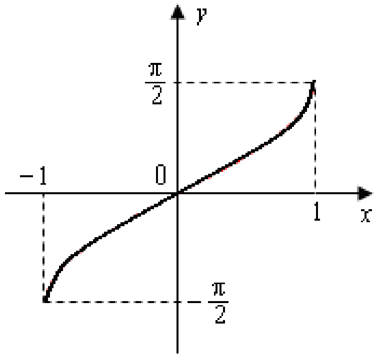
$y = \cos x$



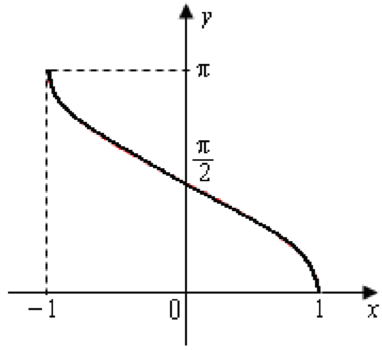
$y = \tan x$



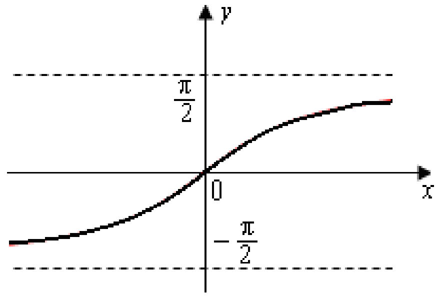
$y = \cot x$



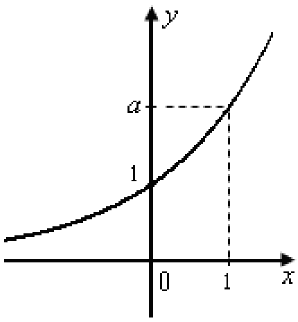
$y = \arcsin\ x$



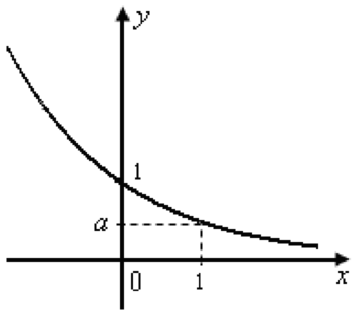
$y = \arccos x$



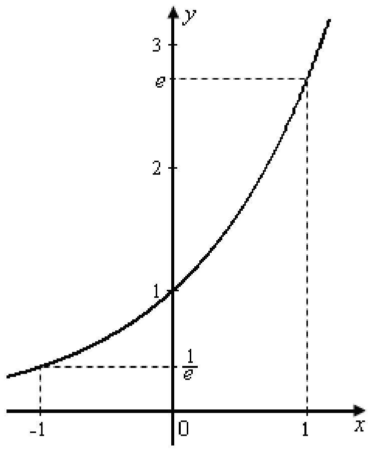
$y = \arctan\ x$



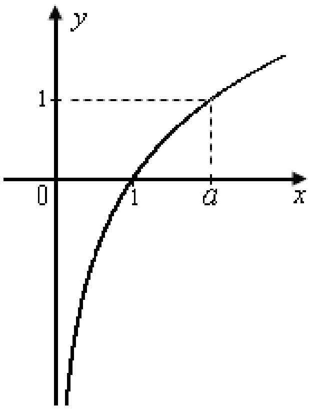
$y = a^x,\ a > 1$



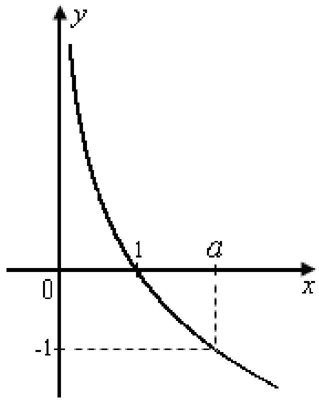
$y = a^x,\ 0 < a < 1$



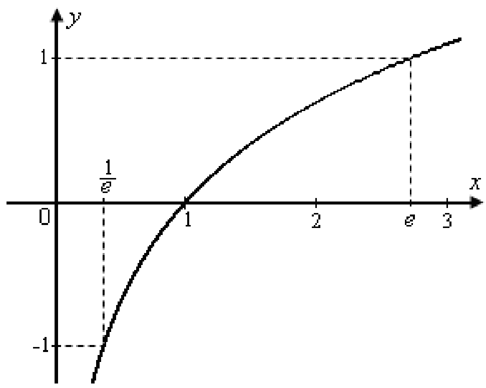
$y = e^x$



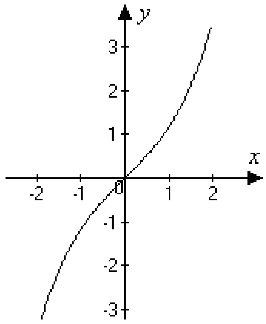
$y = \log_a x,\ a > 1$



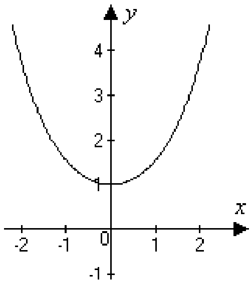
$y = \log_a x,\ 0 < a < 1$



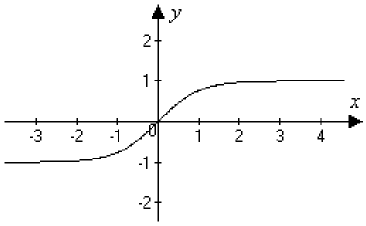
$y = \ln x$



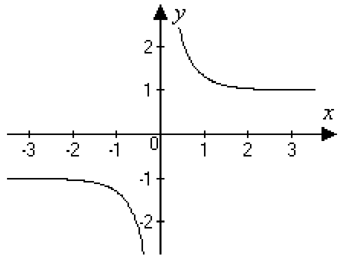
$y = \sinh x$



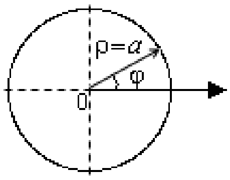
$y = \cosh\ x$



$y = \tanh x$



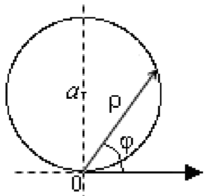
$y = \coth x$



Circle

$$x^2 + y^2 = a^2, \qquad \rho = a$$

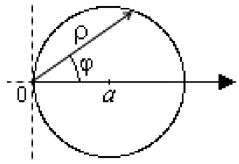
$$\begin{cases} x = a\cos t, \\ y = a\sin t, \end{cases} t \in [0, 2\pi]$$



Circle

$$x^2 + y^2 = 2ay$$

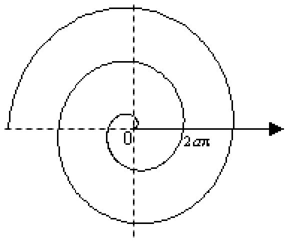
$$\rho = 2a\sin\varphi$$



Circle

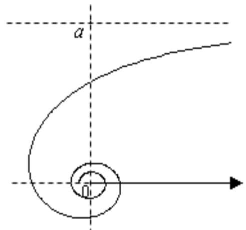
$$x^2 + y^2 = 2ax$$

$$\rho = 2a\cos\varphi$$



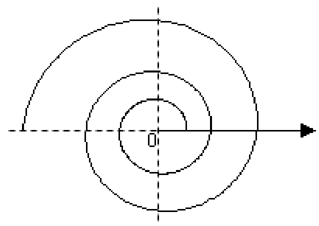
Archimedean spiral

$$\rho = a\varphi$$



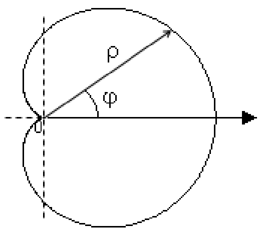
Hyperbolic spiral

$$\rho = \frac{a}{\varphi}$$



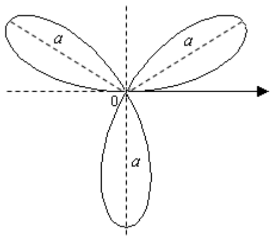
Logarithmic spiral

$$\rho = e^{a\varphi}$$



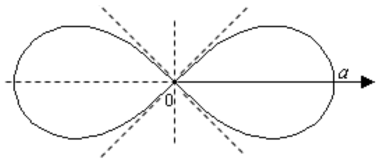
Cardioid

$$\rho = a(1 + \cos\varphi)$$



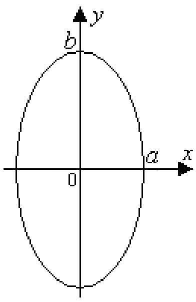
Triple-pentaled rose

$$\rho = a\sin 3\varphi$$



Lemniscate of Bernoulli

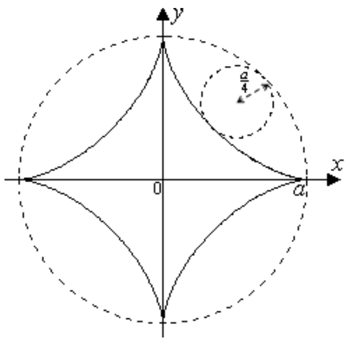
$$\rho = a\sqrt{\cos 2\varphi}$$



Ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

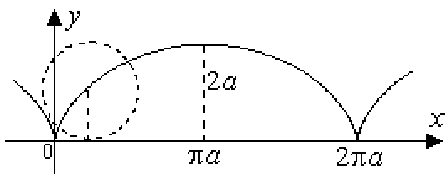
$$\begin{cases} x = a\cos t, \\ y = b\sin t, \end{cases} t \in [0, 2\pi]$$



Astroid

$$x^{2/3} + y^{2/3} = a^{2/3}$$

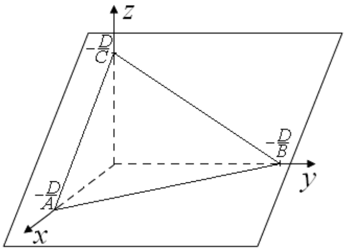
$$\begin{cases} x = a\cos^3 t, \\ y = a\sin^3 t, \end{cases} t \in [0, 2\pi]$$



Cycloid

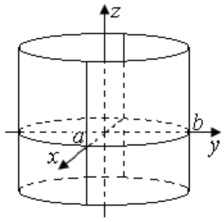
$$\begin{cases} x = a(t - \sin t), \\ y = a(1 - \cos t), \end{cases} t \in [0, 2\pi]$$

Appendix 2. Surfaces in 3D-space



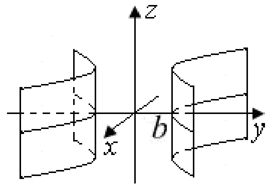
Plane

$Ax + By + Cz + D = 0$



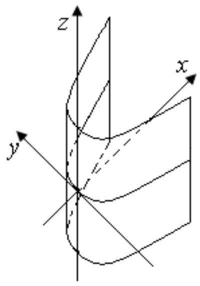
Elliptic Cylinder

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



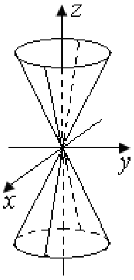
Hyperbolic Cylinder

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$$



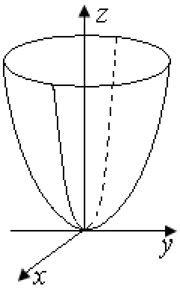
Parabolic Cylinder

$x^2 = 2py, \ (p > 0)$



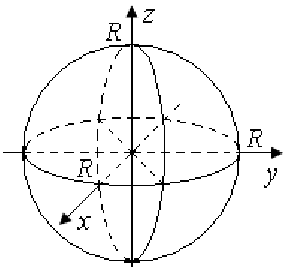
Cone

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$$



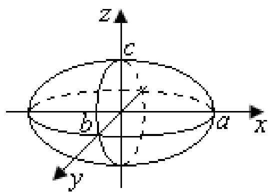
Elliptic Paraboloid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = z$$



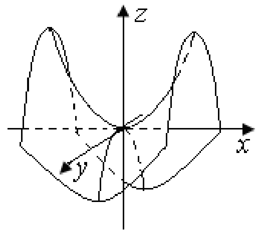
Sphere

$x^2 + y^2 + z^2 = R^2$



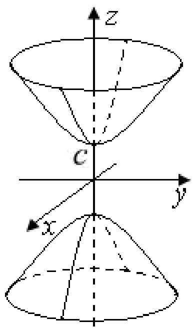
Ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$



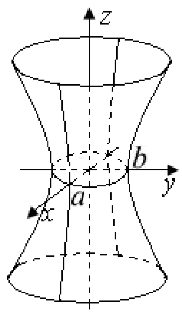
Hyperbolic paraboloid

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = z$$



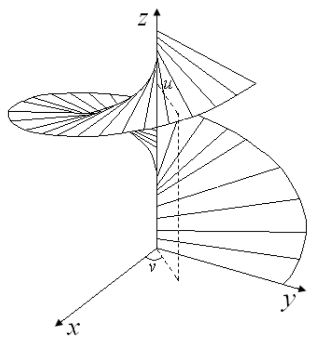
Hyperboloid of Two Sheets

$$\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$



Hyperboloid of One Sheet

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$



Helicoid

$x = u\cos v, \quad y = u\sin v, \quad z = v,$

$0 \le u \le a, \ 0 \le v \le 2\pi.$

Appendix 3. The table of derivatives

$C' = 0 \ \ \forall C \in \mathbb{R};$		$(x)' = 1;$
$(x^n)' = nx^{n-1};$	$\left(\dfrac{1}{x}\right)' = -\dfrac{1}{x^2};$	$\left(\sqrt{x}\right)' = \dfrac{1}{2\sqrt{x}};$
$(e^x)' = e^x;$		$(a^x)' = a^x \ln a;$
$(\ln x)' = \dfrac{1}{x};$		$(\log_a x)' = \dfrac{1}{x \ln a};$
$(\sin x)' = \cos x;$		$(\cos x)' = -\sin x;$
$(\tan x)' = \dfrac{1}{\cos^2 x};$		$(\cot x)' = -\dfrac{1}{\sin^2 x};$
$(\arcsin x)' = \dfrac{1}{\sqrt{1-x^2}};$		$(\arccos x)' = -\dfrac{1}{\sqrt{1-x^2}};$
$(\arctan x)' = \dfrac{1}{1+x^2};$		$(\operatorname{arccot} x)' = -\dfrac{1}{1+x^2};$
$(\sinh x)' = \cosh x;$		$(\cosh x)' = \sinh x;$
$(\tanh x)' = \dfrac{1}{\cosh^2 x};$		$(\coth x)' = -\dfrac{1}{\sinh^2 x};$

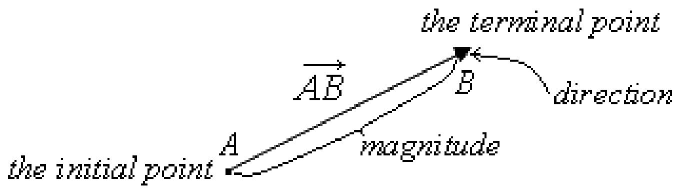
Appendix 4. The Table of Integrals

$\int 0dx = C$	$\int dx = x + C$
$\int x^n dx = \frac{x^{n+1}}{n+1} + C$	$\int \frac{1}{x^2} dx = -\frac{1}{x} + C$
$\int \frac{1}{x} dx = \ln\lvert x\rvert + C$	$\int \frac{1}{2\sqrt{x}} dx = \sqrt{x} + C$
$\int e^x dx = e^x + C$	$\int a^x dx = \frac{a^x}{\ln a} + C$
$\int \sin x dx = -\cos x + C$	$\int \cos x dx = \sin x + C$
$\int \frac{1}{\cos^2 x} dx = \tan x + C$	$\int \frac{1}{\sin^2 x} dx = -\cot x + C$
$\int \sinh x dx = \cosh x + C$	$\int \cosh x dx = \sin x + C$
$\int \frac{1}{\cosh^2 x} dx = \tanh x + C$	$\int \frac{1}{\sinh^2 x} dx = -\coth x + C$
$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \arcsin \frac{x}{a} + C$	$\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \arctan \frac{x}{a} + C$
$\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \ln\left\lvert \frac{x-a}{x+a} \right\rvert + C$	$\int \frac{1}{\sqrt{x^2 \pm a^2}} dx = \ln\left\lvert x + \sqrt{x^2 \pm a^2} \right\rvert + C$

Appendix 5. Vectors and Operations on Vectors

Geometrical Vectors

Vector $\overrightarrow{AB}$ is a line segment having *direction*. This geometric object has *the initial point A* (tail), *the terminal point B* (head) and *the magnitude* (length).



Vector Components

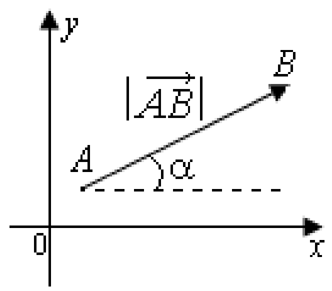
The vector is defined by its coordinates (components). If the coordinates of initial and terminal points are known ($A(x_A, y_A)$ and $B(x_B, y_B)$) then

$$\overrightarrow{AB} = (x_B - x_A, y_B - y_A) = (x_{AB}, y_{AB}) = x_{AB}\vec{i} + y_{AB}\vec{j}.$$

Magnitude and Direction.

The length or magnitude of vector $\overrightarrow{AB} = (x_{AB}, y_{AB})$ is

$$\left|\overrightarrow{AB}\right| = \sqrt{x_{AB}^2 + y_{AB}^2}.$$



The direction is given by the angle between the vector and the positive *x*-axis:

$$\tan\alpha = \frac{y_{AB}}{x_{AB}}.$$

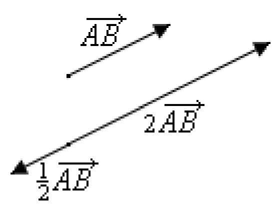
Multiplying a Vector by a Scalar.

The operation of multiplying a vector by a scalar (*scalar multiplication*) is defined as follows

$$k \cdot \overrightarrow{AB} = (k \cdot x_{AB}, k \cdot y_{AB}).$$

If $k>1$ the vector becomes larger, if $0<k<1$ it becomes smaller.

If $k<0$ then vector changes direction on opposite.



Unit vector

The unit vector $\vec{a}_0$ has a magnitude of one $\left(\vec{a}_0 = \dfrac{\vec{a}}{|\vec{a}|}\right)$. These vectors are used to indicate the direction.

Scalar (Dot) Product

Let $\vec{a}$ and $\vec{b}$ be two vectors and ψ be the angle between them. *The scalar* (*dot*) *product* of $\vec{a}$ and $\vec{b}$ is defined

$$\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot \cos\psi = a_x b_x + a_y b_y + a_z b_z.$$

The *projection* of $\vec{b}$ onto $\vec{a}$

$$\mathrm{pr}_{\vec{a}} \vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}.$$

Vector (Cross) Product

The vector (*cross*) *product* of $\vec{a}$ and $\vec{b}$ is another vector which is perpendicular to these two vectors.

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \vec{i} \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} - \vec{j} \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} + \vec{k} \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix}.$$

Scalar Triple Product

The scalar triple product (mixed product) of vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is defined as

$$\vec{a}\vec{b}\vec{c} = \vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix};$$

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